

A NONCOMMUTATIVE DE FINETTI THEOREM FOR BOOLEAN INDEPENDENCE

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ABSTRACT. We introduce a family of quantum semigroups and its natural coactions on noncommutative polynomials. We define three invariance conditions for the joint distribution of sequences of selfadjoint noncommutative random variables associated with these coactions. For one of the invariance conditions, we show that the joint distribution of an infinite sequence of noncommutative random variables satisfies it is equivalent to the fact that the sequence of the random variables is identically distributed and boolean independent with respect to the conditional expectation onto its tail algebra. This is a boolean analogue of de Finetti theorem on exchangeable sequences. In the end of the paper, we also discuss the other two invariance conditions which lead to some trivial results.

1. INTRODUCTION

In classical probability, the study of random variables with distributional symmetries was started by the pioneering work of de Finetti on 2-point valued random variables. One of the most general versions of de Finetti's work states that an infinite sequence of random variables, whose joint distribution is invariant under all finite permutations, is conditionally independent and identically distributed. One can see e.g. [12] for an exposition on the classical de Finetti theorem for more details. Also, see [11], Hewitt and Savage considered the distributional symmetries of random variables which are distributed on $X = E \times E \times E \times \cdots$, where E is a compact Hausdorff space. Later, in [21], an early noncommutative version of de Finetti theorem was given by Størmer. His work focused on exchangeable states on the infinite reduced tensor product of C^* -algebras. Roughly speaking, in noncommutative probability theory, Størmer studied symmetric states on commuting noncommutative random variables. Recently, in [14], without the commuting relation, Köstler studied exchangeable sequences of noncommutative random variables in W^* -probability spaces with normal faithful states. In classical probability, if the second moment of a real valued random variable is 0, then the random variable is 0 a.e.. Faithfulness is a natural generalization of this property in noncommutative probability, readers are referred to [24]. Köstler showed that exchangeable sequences of random variables possess some kind of factorization property, but the exchangeability does not imply any kind of universal relation. In other words, we can not expect to determine mixed moments of an exchangeable sequence of random variables in Speicher's universal sense [19]. By strengthening "exchangeability" to invariance under a coaction of the free quantum permutations, in [15], Köstler and Speicher discovered that the de Finetti theorem has a natural analogue in Voiculescu's free probability theory (see [24]). Here, free quantum permutations refer to Wang's quantum groups $\mathcal{A}_s(n)$ in [27].

Köstler and Speicher's work starts a systematic study of the probabilistic symmetries on noncommutative probability theory. Most of the further projects are developed by Banica, Curran and Speicher, see [1],[5],[6]. They showed their de Finetti type theorems in both of the classical(commutative) probability theory and the noncommutative probability theory under the invariance conditions of easy groups and easy quantum groups, respectively. All these works in noncommutative case are proceeded under the assumption that the state of the probability space is faithful. This is a natural assumption in free probability theory, because in [7], Dykema showed that the free product of a family of W^* -probability spaces with normal faithful states is also a W^* -probability space with a normal faithful state. Thus the category of W^* -probability spaces with faithful states is closed under the free product construction. Since the family of W^* -probability spaces with normal faithful states is a part of W^* -probability spaces with normal faithful states, one may ask what happens to probability spaces with states which are not necessarily faithful. More specific, what is the de Finetti type theorem for more general noncommutative probability spaces?

Recall that in the noncommutative realm, besides the freeness and the classical independence, there are many other kinds of independence relations, e.g. monotone independence [16], boolean independence[20], type B independence [3] and more recently two-face freeness for pairs of random variables[23]. All these types of independence are associated with certain products on probability spaces. Among these products, in [19], Speicher showed that there are only two universal products on the unital noncommutative probability spaces, namely the tensor product and the free product. The corresponding independent relations associated with these two universal products are the classical independence and the free independence. It was also showed in [19] that there is a unique universal product in the non-unital framework which is called boolean product. This non-unital universal product provides a way to construct probability spaces with non-faithful states from probability spaces with faithful states. The more general noncommutative probability spaces will be defined in section 6 which are called noncommutative probability spaces with non-degenerated states. We would expect that boolean independence plays the same role in noncommutative probability spaces with non-degenerated states as the classical independence and the freeness play in commutative probability spaces and noncommutative probability spaces with faithful states, respectively. The main purpose of this work is to give certain distributional symmetries which can characterize conditionally boolean independence in de Finetti theorem's form.

To proceed this work, we will construct a class of quantum semigroups $B_s(n)$'s and its sub quantum semigroups $\mathcal{B}_s(n)$'s. Then, we can define a coaction of $B_s(n)$ on the set of noncommutative polynomials with n -indeterminants. Unlike $B_s(n)$, there are two natural ways to define coactions of $\mathcal{B}_s(n)$ on the set of noncommutative polynomials. The first way considers the set of noncommutative polynomials as a linear space, the coaction of $\mathcal{B}_s(n)$ defined on the linear space will be called the linear coaction of $\mathcal{B}_s(n)$ on the set of noncommutative polynomials. The second way defines the coaction of $\mathcal{B}_s(n)$ by considering the set of noncommutative polynomials as an algebra, the coaction of $\mathcal{B}_s(n)$ defined as a coaction on the algebra will be called the algebraic coaction of $\mathcal{B}_s(n)$ on the set of noncommutative polynomials. With these three coactions of the quantum semigroups on the set of noncommutative polynomials with n -indeterminants, we can describe three invariance conditions for the joint distribution of any sequence of

n random variables (x_1, \dots, x_n) . We will show that the invariance conditions determined by the algebraic coaction of $\mathcal{B}_s(n)$ and the coaction of $B_s(n)$ are so strong such that if the joint distribution of the sequence of n random variables (x_1, \dots, x_n) satisfies one of the invariance conditions, then $x_1 = x_2 = \dots = x_n$ or $x_1 = x_2 = \dots = x_n = 0$, respectively. In this paper, we are mainly concerned with the invariance conditions which are determined by the linear coactions of the quantum semigroups $\mathcal{B}_s(n)$'s. Before proving the main theorems, we will study tail algebras in W^* probability spaces with non-degenerated normal states. There will be a brief discussion on why we should consider these more general spaces. Unlike W^* probability spaces with faithful normal states, we will define two kinds of tail algebras, one contains the unit of the original algebra and the other may not. As Köstler did in [14], we will define our conditional expectation by taking the WOT limit of "shifts". One of the differences between our work and Köstler' result is that our tail algebra may not contain the unit of the original algebra. Then, we will prove the following theorem for the two different cases (tail algebra with the unit of the original algebra or not):

Theorem 1.1. *Let (\mathcal{A}, ϕ) be a W^* -probability space and $(x_i)_{i \in \mathbb{N}}$ be an infinite sequence of selfadjoint random variables which generate \mathcal{A} as a von Neumann algebra and the unit of \mathcal{A} is (not) contained in the WOT closure of the non unital algebra generated by $(x_i)_{i \in \mathbb{N}}$. Then the following are equivalent:*

- a) *The joint distribution of $(x_i)_{i \in \mathbb{N}}$ satisfies the invariance condition associated with the linear coactions of the quantum semigroups $\mathcal{B}_s(n)$'s.*
- b) *The sequence $(x_i)_{i \in \mathbb{N}}$ is identically distributed and boolean independent with respect to the ϕ -preserving conditional expectation E onto the non unital(unital) tail algebra of the $(x_i)_{i \in \mathbb{N}}$*

One can see the definitions of $A_s(n)$ and $\mathcal{B}_s(n)$ in section 2 and 3 for details. It should be mentioned here that Wang's quantum permutation group $A_s(n)$ is a quotient algebra of $\mathcal{B}_s(n)$ for each n . Moreover, both of the invariance conditions associated with the linear coactions and the algebraic coactions of the quantum semigroups $\mathcal{B}_s(n)$'s are stronger than the invariance condition associated with the quantum permutations $A_s(n)$'s

The paper is organized as follows: In Section 2, we recall the basic definitions and notation from the noncommutative probability theory, Wang's quantum groups and exchangeable sequence of random variables. In Section 3, we introduce our quantum semigroup $B_s(n)$ and its sub quantum semigroups $\mathcal{B}_s(n)$. Then, we introduce a linear coaction of the quantum semigroup $\mathcal{B}_s(n)$ on the set of the noncommutative polynomials. We will define an invariance condition associated with the linear coaction of $\mathcal{B}_s(n)$. In section 4, we have a brief discussion on the relation between freeness and boolean independence. We show that operator valued boolean independence implies operator valued freeness in some special cases. In section 5, we prove that the joint distribution of a finite sequence of n boolean independent operator valued random variables are invariant under the linear coaction of $\mathcal{B}_s(n)$. In section 6, we recall the properties of the tail algebra of any infinite exchangeable sequences of noncommutative variables and study the properties of the tail algebra under the boolean exchangeable condition. In section 7, we will prove the main theorems and provide some examples. In section 8, we define a coaction of $B_s(n)$ and an algebraic coaction of $\mathcal{B}_s(n)$ on the set of noncommutative polynomials in n indeterminants. Then, we define the invariance conditions

associated with these coactions. We will study the set of random variables (x_1, \dots, x_n) whose joint distribution satisfies one of these invariance conditions.

2. PRELIMINARIES AND NOTATION

2.1. Noncommutative probability space. We recall some necessary definitions and notation of noncommutative probability spaces. For further details, see texts [15], [17], [2], [24].

Definition 2.1. A non-commutative probability space (\mathcal{A}, ϕ) consists of a unital algebra \mathcal{A} and a linear functional $\phi : \mathcal{A} \rightarrow \mathbb{C}$. (\mathcal{A}, ϕ) is called a $*$ -probability space if \mathcal{A} is a $*$ -algebra and $\phi(xx^*) \geq 0$ for all $x \in \mathcal{A}$. (\mathcal{A}, ϕ) is called a W^* -probability space if \mathcal{A} is a W^* -algebra and ϕ is a normal state on it. We say (\mathcal{A}, ϕ) is tracial if

$$\phi(xy) = \phi(yx), \quad \forall x, y \in \mathcal{A}.$$

The elements of \mathcal{A} are called random variables. Let $x \in \mathcal{A}$ be a random variable, then its distribution is a linear functional μ_x on $\mathbb{C}[X]$ (the algebra of complex polynomials in one variable), defined by $\mu_x(P) = \phi(P(x))$.

Note that we do not require the state on W^* -probability space to be tracial. We will specify the probability spaces we concern in section 6 and section 8.

Definition 2.2. Let I be an index set, the algebra of noncommutative polynomials in $|I|$ variables, $\mathbb{C}\langle X_i | i \in I \rangle$, is the linear span of 1 and noncommutative monomials of the form $X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}$ with $i_1 \neq i_2 \neq \cdots \neq i_n \in I$ and all k_j 's are positive integers. For convenience we will use $\mathbb{C}\langle X_i | i \in I \rangle_0$ to denote the set of noncommutative polynomials without constant term.

Let $(x_i)_{i \in I}$ be a family of random variables in a noncommutative probability space (\mathcal{A}, ϕ) . Their joint distribution is a linear functional $\mu : \mathbb{C}\langle X_i | i \in I \rangle \rightarrow \mathbb{C}$ defined by

$$\mu(X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}),$$

and $\mu(1) = 1$.

Remark 2.3. In general, the joint distribution depends on the order of the random variables, e.g. let $I = \{1, 2\}$, then μ_{x_1, x_2} may not equal μ_{x_2, x_1} . According to our notation, $\mu_{x_1, x_2}(X_1 X_2) = \phi(x_1 x_2)$, but $\mu_{x_2, x_1}(X_1 X_2) = \phi(x_2 x_1)$.

Definition 2.4. Let (\mathcal{A}, ϕ) be a noncommutative probability space, a family of unital subalgebras $(\mathcal{A}_i)_{i \in I}$ is said to be free if

$$\phi(a_1 \cdots a_n) = 0,$$

whenever $a_k \in \mathcal{A}_{i_k}$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\phi(a_k) = 0$ for all k . Let $(x_i)_{i \in I}$ be a family of random variables and \mathcal{A}_i 's be the unital subalgebras generated by x_i 's, respectively. We say the family of random variables $(x_i)_{i \in I}$ is free if the family of unital subalgebras $(\mathcal{A}_i)_{i \in I}$ is free.

Definition 2.5. Let (\mathcal{A}, ϕ) be a noncommutative probability space, a family of (usually non-unital) subalgebras $\{\mathcal{A}_i | i \in I\}$ of \mathcal{A} is said to be boolean independent if

$$\phi(x_1 x_2 \cdots x_n) = \phi(x_1) \phi(x_2) \cdots \phi(x_n)$$

whenever $x_k \in \mathcal{A}_{i(k)}$ with $i(1) \neq i(2) \neq \dots \neq i(n)$. A set of random variables $\{x_i \in \mathcal{A} \mid i \in I\}$ is said to be boolean independent if the family of non-unital subalgebras \mathcal{A}_i , which are generated by x_i respectively, is boolean independent.

One refers to [9] for more details of boolean product of random variables. Since the framework for boolean independence is a non-unital algebra in general, we will not require our operator valued probability spaces to be unital:

Definition 2.6. An operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \rightarrow \mathcal{B})$ consists of an algebra \mathcal{A} , a subalgebra \mathcal{B} of \mathcal{A} and a $\mathcal{B} - \mathcal{B}$ bimodule linear map $E : \mathcal{A} \rightarrow \mathcal{B}$ i.e.

$$E[b_1 a b_2] = b_1 E[a] b_2,$$

for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$. According to the definition in [22], we call E a conditional expectation from \mathcal{A} to \mathcal{B} if E is onto, i.e. $E[\mathcal{A}] = \mathcal{B}$. The elements of \mathcal{A} are called random variables.

In operator valued free probability theory, \mathcal{A} and \mathcal{B} are unital and have the same unit

Definition 2.7. Given an algebra \mathcal{B} , we denote by $\mathcal{B}\langle X \rangle$ the algebra which is freely generated by \mathcal{B} and the indeterminant X . Let 1_X be the identity of $\mathbb{C}\langle X \rangle$, then $\mathcal{B}\langle X \rangle$ is set of linear combinations of the elements in \mathcal{B} and the noncommutative monomials $b_0 X b_1 X b_2 \dots b_{n-1} X b_n$ where $b_k \in \mathcal{B} \cup \{\mathbb{C}1_X\}$ and $n \geq 1$. The elements in $\mathcal{B}\langle X \rangle$ are called \mathcal{B} -polynomials. In addition, $\mathcal{B}\langle X \rangle_0$ denotes the subalgebra of $\mathcal{B}\langle X \rangle$ which doesn't contain the constant term i.e. the linear span of the noncommutative monomials $b_0 X b_1 X b_2 \dots b_{n-1} X b_n$ where $b_k \in \mathcal{B} \cup \{\mathbb{C}1_X\}$ and $n \geq 1$. $\mathcal{B}\langle X \rangle_0$.

Given an operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \rightarrow \mathcal{B})$ such that \mathcal{A} and \mathcal{B} are unital. A family of unital subalgebras $\{\mathcal{A}_i \supset \mathcal{B}\}_{i \in I}$ is said to be freely independent with respect to E if

$$E[a_1 \dots a_n] = 0,$$

whenever $i_1 \neq i_2 \neq \dots \neq i_n$, $a_k \in \mathcal{A}_{i_k}$ and $E[a_k] = 0$ for all k . A family of $(x_i)_{i \in I}$ is said to be free independent over \mathcal{B} , if the unital subalgebras $\{\mathcal{A}_i\}_{i \in I}$ which are generated by x_i and \mathcal{B} respectively is free, or equivalently

$$E[p_1(x_{i_1}) p_2(x_{i_2}) \dots p_n(x_{i_n})] = 0,$$

whenever $i_1 \neq i_2 \neq \dots \neq i_n$, $p_1, \dots, p_n \in \mathcal{B}\langle X \rangle$ and $E[p_k(x_{i_k})] = 0$ for all k .

Let $\{x_i\}_{i \in I}$ be a family of random variables in an operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \rightarrow \mathcal{B})$. \mathcal{A}, \mathcal{B} are not necessarily unital. $\{x_i\}_{i \in I}$ is said to be boolean independent over \mathcal{B} if for all $i_1, \dots, i_n \in I$, with $i_1 \neq i_2 \neq \dots \neq i_n$ and all \mathcal{B} -valued polynomials $p_1, \dots, p_n \in \mathcal{B}\langle X \rangle_0$ such that

$$E[p_1(x_{i_1}) p_2(x_{i_2}) \dots p_n(x_{i_n})] = E[p_1(x_{i_1})] E[p_2(x_{i_2})] \dots E[p_n(x_{i_n})].$$

2.2. Wang's quantum permutation groups. In [27], Wang introduced the following quantum groups $A_s(n)$'s.

Definition 2.8. $A_s(n)$ is defined as the universal unital C^* -algebra generated by elements u_{ij} ($i, j = 1, \dots, n$) such that we have

- each u_{ij} is an orthogonal projection, i.e. $u_{ij}^* = u_{ij} = u_{ij}^2$ for all $i, j = 1, \dots, n$.

- the elements in each row and column of $u = (u_{ij})_{i,j=1,\dots,n}$ form a partition of unit, i.e. are orthogonal and sum up to 1: for each $i = 1, \dots, n$ and $k \neq l$ we have

$$u_{ik}u_{il} = 0 \quad \text{and} \quad u_{ki}u_{li} = 0;$$

and for each $i = 1, \dots, n$ we have

$$\sum_{k=1}^n u_{ik} = 1 = \sum_{k=1}^n u_{ki}.$$

$A_s(n)$ is a compact quantum group in the sense of Woronowicz [26], with comultiplication, counit and antipode given by the formulas:

$$\begin{aligned} \Delta u_{ij} &= \sum_{k=1}^n u_{ik} \otimes u_{kj} \\ \epsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji}. \end{aligned}$$

The right coaction of $A_s(n)$ on $\mathbb{C}\langle X_1, \dots, X_n \rangle$ is a linear map $\alpha : \mathbb{C}\langle X_1, \dots, X_n \rangle \rightarrow \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes A_s(n)$ given by:

$$\alpha(X_{i_1}X_{i_2}\cdots X_{i_m}) = \sum_{j_1, \dots, j_m=1}^n X_{j_1}X_{j_2}\cdots X_{j_m} \otimes u_{j_1, i_1}u_{j_2, i_2}\cdots u_{j_m, i_m},$$

where \otimes denotes the algebraic tensor product.

In the earlier papers, α is defined as an algebraic homomorphism. We emphasis on the linearity here because we will define some coactions of our quantum semigroups on noncommutative polynomials in a similar way. The right coaction has the following property:

$$(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha.$$

Let $(x_i)_{i \in \mathbb{N}}$ be an infinite sequence of random variables in a noncommutative probability space (\mathcal{A}, ϕ) , the sequence is said to be quantum exchangeable if their joint distribution is invariant under Wang's quantum permutation groups, i.e. for all n , we have

$$\mu_{x_1, \dots, x_n}(p)1_{A_s(n)} = \mu_{x_1, \dots, x_n} \otimes id_{A_s(n)}(\alpha(p)),$$

where μ_{x_1, \dots, x_n} is the joint distribution of x_1, \dots, x_n with respect to ϕ and $p \in \mathbb{C}\langle X_1, \dots, X_n \rangle$. For example, if $p = X_{i_1}X_{i_2}\cdots X_{i_m}$, then the equation above can be written as:

$$\begin{aligned} \phi(x_{i_1}x_{i_2}\cdots x_{i_m})1_{A_s(n)} &= \mu_{x_1, \dots, x_n}((X_{i_1}X_{i_2}\cdots X_{i_m})1_{A_s(n)}) \\ &= \mu_{x_1, \dots, x_n} \otimes id_{A_s(n)}\left(\sum_{j_1, \dots, j_m=1}^n X_{j_1}X_{j_2}\cdots X_{j_m} \otimes u_{j_1, i_1}u_{j_2, i_2}\cdots u_{j_m, i_m}\right) \\ &= \sum_{j_1, \dots, j_m=1}^n \phi(x_{j_1}x_{j_2}\cdots x_{j_m})u_{j_1, i_1}u_{j_2, i_2}\cdots u_{j_m, i_m}, \end{aligned}$$

whenever $i_1 \neq i_2 \neq \dots \neq i_n$.

Let S_n be the permutation group on $\{1, \dots, n\}$. The joint distribution of $(x_i)_{i \in \mathbb{N}}$ is said to be exchangeable if for all n , $\sigma \in S_n$, we have

$$\mu_{x_1, \dots, x_n} = \mu_{x_{\sigma(1)}, \dots, x_{\sigma(n)}},$$

where μ_{x_1, \dots, x_n} is the joint distribution of x_1, \dots, x_n with respect to ϕ . It is showed in [15] that quantum exchangeability implies classical exchangeability.

3. QUANTUM SEMIGROUPS

Our probabilistic symmetries will be given by the invariance conditions associated with certain coactions of our quantum semigroups. First, we recall the related definitions and notation of quantum semigroups.

A quantum space is an object of the category dual to the category of C^* -algebras ([25]). For any C^* -algebras A and B , the set of morphisms $\text{Mor}(A, B)$ consists of all C^* -algebra homomorphisms acting from A to $M(B)$, where $M(B)$ is the multiplier algebra of B , such that $\phi(A)B$ is dense in B . If A and B are unital C^* -algebras, then all unital C^* -homomorphisms from A to B are in $\text{Mor}(A, B)$. In [18],

Definition 3.1. *By a quantum semigroup we mean a C^* -algebra \mathcal{A} endowed with an additional structure described by a morphism $\Delta \in \text{Mor}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ such that*

$$(\Delta \otimes id_{\mathcal{A}})\Delta = (id_{\mathcal{A}} \otimes \Delta)\Delta.$$

In other words, Δ defines a comultiplication on \mathcal{A} . Here the tensor product \otimes denotes the minimal tensor product \otimes_{min} .

Now, we turn to introduce our quantum semigroups:

Quantum semigroups $(B_s(n), \Delta)$: The algebra $B_s(n)$ is defined as the universal unital C^* -algebra generated by elements $u_{i,j}$ ($i, j = 1, \dots, n$) and a projection \mathbf{P} such that we have

- each $u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, \dots, n$,
-

$$u_{i,k}u_{i,l} = 0 \quad \text{and} \quad u_{k,i}u_{l,i} = 0,$$

whenever $k \neq l$

- For all $1 \leq i \leq n$, $\mathbf{P} = \sum_{k=1}^n u_{k,i} \mathbf{P}$.

We will denote the unite of $B_s(n)$ by I , the projection \mathbf{P} is called the invariant projection of $B_s(n)$.

On this unital C^* -algebra, we can define a unital C^* -homomorphism

$$\Delta : B_s(n) \rightarrow B_s(n) \otimes B_s(n)$$

by the following formulas:

$$\Delta u_{i,j} = \sum_{k=1}^n u_{i,k} \otimes u_{k,j}$$

and

$$\Delta \mathbf{P} = \mathbf{P} \otimes \mathbf{P}, \quad \Delta I = I \otimes I.$$

We will see that $(B_s(n), \Delta)$ is a quantum semigroup. To show this we need to check that Δ defines a unital C^* -homomorphism from $B_s(n)$ to $B_s(n) \otimes B_s(n)$ and satisfies the comultiplication condition :

First, $\Delta u_{i,j} = \sum_{k=1}^n u_{i,k} \otimes u_{k,j}$ is a projection because $u_{i,k}, u_{k,j}$ are projections and $u_{i,k}u_{i,l} = 0$ if $k \neq l$, $u_{i,k} \otimes u_{k,j}$'s are orthogonal to each other. Also, $\Delta \mathbf{P} = \mathbf{P} \otimes \mathbf{P}$ is a projection.

Let $l \neq m$, then

$$\begin{aligned}
\Delta(u_{i,l})\Delta u_{i,m} &= \left(\sum_{k=1}^n u_{i,k} \otimes u_{k,l}\right) \left(\sum_{j=1}^n u_{i,j} \otimes u_{j,m}\right) \\
&= \sum_{k,j=1}^n u_{i,k} u_{i,j} \otimes u_{k,l} u_{j,m} \\
&= \sum_{k=1}^n u_{i,k} \otimes u_{k,l} u_{k,m} \\
&= 0.
\end{aligned}$$

The same, we have $\Delta(u_{l,i})\Delta u_{m,i} = 0$, for $m \neq l$. Moreover, we have

$$\begin{aligned}
\Delta\left(\sum_{l=1}^n u_{l,i}\right)\Delta \mathbf{P} &= \left(\sum_{l,k=1}^n u_{l,k} \otimes u_{k,i}\right) \mathbf{P} \otimes \mathbf{P} \\
&= \sum_{l,k=1}^n u_{l,k} \mathbf{P} \otimes u_{k,i} \mathbf{P} \\
&= \sum_{k=1}^n \mathbf{P} \otimes u_{k,i} \mathbf{P} \\
&= \mathbf{P} \otimes \mathbf{P}.
\end{aligned}$$

and Δ sends the unit of $B_s(n)$ to the unit of $B_s(n) \otimes B_s(n)$. Therefore, Δ defines a unital C^* -homomorphism on $B_s(n)$ by the universality of $B_s(n)$.

The comultiplication condition holds, because on the generators we have:

$$(\Delta \otimes id_{\mathcal{A}})\Delta u_{i,j} = \sum_{k,l=1}^n u_{i,k} \otimes u_{k,l} \otimes u_{l,j} = (id_{\mathcal{A}} \otimes \Delta)\Delta u_{i,j}$$

$$(\Delta \otimes id_{\mathcal{A}})\Delta \mathbf{P} = \mathbf{P} \otimes \mathbf{P} \otimes \mathbf{P} = (id_{\mathcal{A}} \otimes \Delta)\Delta \mathbf{P}$$

$$(\Delta \otimes id_{\mathcal{A}})\Delta I = I \otimes I \otimes I = (id_{\mathcal{A}} \otimes \Delta)\Delta I.$$

Therefore, $(B_s(n), \Delta)$ is a quantum semigroup.

Remark 3.2. *If we let the invariant projection to be the identity, then we get Wang's free quantum permutation group. Therefore, $A_s(n)$ is a quotient C^* -algebra of $B_s(n)$, i.e. there exists a unital C^* -homomorphism $\beta : B_s(n) \rightarrow A_s(n)$ such that β is surjective.*

Now, we provide some nontrivial representations of $B_s(n)$'s:

Let \mathbb{C}^6 be the standard 6-dimensional complex Hilbert space with orthonormal basis v_1, \dots, v_6 . Let

$$\begin{aligned}
P_{11} &= P_{v_1+v_2}, & P_{21} &= P_{v_3+v_4}, & P_{13} &= P_{v_5+v_6}, \\
P_{21} &= P_{v_3+v_6}, & P_{22} &= P_{v_5+v_2}, & P_{23} &= P_{v_1+v_4}, \\
P_{31} &= P_{v_4+v_5}, & P_{32} &= P_{v_1+v_6}, & P_{33} &= P_{v_2+v_3}.
\end{aligned}$$

and $P = P_{v_1+v_2+v_3+v_4+v_5+v_6}$, where P_v denotes the one dimensional orthogonal projection onto the subspace spanned by v . Then the unital algebra generated by $P_{i,j}$ and P gives a representation π of $B_s(3)$ on \mathbb{C}^6 by the following formulas on the generators of $\mathcal{B}_s(3)$:

$$\pi(I) = I_{\mathbb{C}^6}, \quad \pi(u_{i,j}) = P_{ij}, \quad \pi(\mathbf{P}) = P.$$

π is well defined by the universality of $B_s(3)$.

Moreover, the matrix form for $P_{1,1}$ and P with respect to the basis are

$$P_{11} = 1/2 \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } P = 1/6 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

then we have

$$PP_{11}P = 1/18 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1/3P.$$

In general, we have

Lemma 3.3. *Let v_1, \dots, v_{2n} be an orthonormal basis of the standard $2n$ -dimensional Hilbert space \mathbb{C}^{2n} , and let $v_k = v_{k+2n}$ for all $k \in \mathbb{Z}$, let*

$$P_{i,j} = P_{v_{2(i-j)+1} + v_{2(j-i)+2}},$$

where P_v is the orthogonal projection the one dimensional subspace generated by the vector v and $P = P_{v_1+v_2+\dots+v_{2n}}$, $\mathbf{1}$ is the identity of $B(\mathbb{C}^{2n})$. Then $\{P_{i,j}\}_{i,j=1,\dots,n}$, P and $\mathbf{1}$ satisfy the defining conditions of the algebra $B_s(n)$,

Proof. It is easy to see that the inner product

$$\langle v_{2(i-j)+1} + v_{2(j-i)+2}, v_{2(i-k)+1} + v_{2(k-i)+2} \rangle = 2\delta_{j,k},$$

so $P_{ik}P_{ij} = 0$ if $j \neq k$. The same $P_{ki}P_{ji} = 0$ if $k \neq j$. Fix i , we see that $v_1 + v_2 + \dots + v_{2n} \in \text{span}\{v_{2(i-j)+1} + v_{2(j-i)+2} | j = 1, \dots, n\}$, so $\sum_{k=1}^n P_{ik}P = P$. \square

Therefore, by lemma 3.3, there exists a representation π of $B_s(n)$ on \mathbb{C}^{2n} which is defined by the following formulas:

$$\pi(\mathbf{1}_{B_s(n)}) = \mathbf{1}, \quad \pi(\mathbf{P}) = P$$

and

$$\pi(u_{i,j}) = P_{i,j},$$

for all $i, j = 1, \dots, n$.

Now, we turn to introduce a sub quantum semigroup of $(B_s(n), \Delta)$. Since $\mathbf{P} \neq I$ is a projection in $B_s(n)$, $\mathcal{B}_s(n) = \mathbf{P}B_s(n)\mathbf{P}$ is a C^* -algebra with identity \mathbf{P} and generators

$$\{\mathbf{P}u_{i_1,j_1} \cdots u_{i_k,j_k}\mathbf{P} | i_1, j_1, \dots, i_k, j_k \in \{1, \dots, n\}, k \geq 0\}.$$

If we restrict the comultiplication Δ onto $\mathcal{B}_s(n)$, then we have

$$\Delta(\mathbf{P}u_{i_1,j_1} \cdots u_{i_k,j_k}\mathbf{P}) = (\mathbf{P} \otimes \mathbf{P}) \left(\sum_{l_1, \dots, l_k=1}^n u_{i_1,l_1} \cdots u_{i_k,l_k} \otimes u_{l_1,j_1} \cdots u_{l_k,j_k} \right) (\mathbf{P} \otimes \mathbf{P}),$$

which is contained in $\mathcal{B}_s(n) \otimes \mathcal{B}_s(n)$. Therefore, $(\mathcal{B}_s(n), \Delta)$ is also a quantum semigroup and \mathbf{P} is the identity of $\mathcal{B}_s(n)$. We will call $\mathcal{B}_s(n)$ the boolean permutation quantum semigroup of n .

Remark 3.4. *If we require $\mathbf{P}u_{i,j} = u_{i,j}\mathbf{P}$ for all $i, j = 1, \dots, n$, then the universal algebra we constructed in the above way is exactly Wang's quantum permutation group. Therefore, $A_s(n)$ is also a quotient algebra of $\mathcal{B}_s(n)$.*

In the following definition, \otimes denotes the tensor product for linear spaces:

Definition 3.5. *Let $\mathcal{S} = (\mathcal{A}, \Delta)$ be a quantum semigroup and \mathcal{V} be a complex vector space, by a (right) coaction of the quantum group \mathcal{S} on \mathcal{V} we mean a linear map $L : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{A}$ such that*

$$(L \otimes id)L = (id \otimes \Delta)L.$$

We say a linear functional $\omega : \mathcal{V} \rightarrow \mathbb{C}$ is invariant under L if

$$(\omega \otimes id)L(v) = \omega(v)I_{\mathcal{A}},$$

where $I_{\mathcal{A}}$ is the identity of \mathcal{A} .

Given a complex vector space \mathcal{W} , We say a linear map $T : \mathcal{V} \rightarrow \mathcal{W}$ is invariant under L if

$$(T \otimes id)L(v) = T(v) \otimes I_{\mathcal{A}}.$$

Remark 3.6. *This definition is about coactions on linear spaces but not coactions on algebras.*

Let $\mathbb{C}\langle X_1, \dots, X_n \rangle$ be the set of noncommutative polynomials in n indeterminants, which is a linear space over \mathbb{C} with basis $X_{i_1} \cdots X_{i_k}$ for all integer $k \geq 0$ and $i_1, \dots, i_k \in \{1, \dots, n\}$.

Now, we define a right coaction L_n of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \dots, X_n \rangle$ as follows:

$$L_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1, \dots, j_k=1}^n X_{j_1} \cdots X_{j_k} \otimes \mathbf{P}u_{j_1, i_1} \cdots u_{j_n, i_n} \mathbf{P}.$$

It is a well defined coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \dots, X_n \rangle$, because:

$$\begin{aligned} & (L_n \otimes id)L_n(X_{i_1} \cdots X_{i_k}) \\ &= (L_n \otimes id) \sum_{j_1, \dots, j_k=1}^n X_{j_1} \cdots X_{j_k} \otimes \mathbf{P}u_{j_1, i_1} \cdots u_{j_n, i_n} \mathbf{P} \\ &= \sum_{j_1, \dots, j_k=1}^n \sum_{l_1, \dots, l_k=1}^n X_{l_1} \cdots X_{l_k} \otimes \mathbf{P}u_{l_1, j_1} \cdots u_{l_n, j_n} \mathbf{P} \otimes \mathbf{P}u_{j_1, i_1} \cdots u_{j_n, i_n} \mathbf{P} \\ &= \sum_{l_1, \dots, l_k=1}^n X_{l_1} \cdots X_{l_k} \otimes \left(\sum_{j_1, \dots, j_k=1}^n \mathbf{P}u_{l_1, j_1} \cdots u_{l_n, j_n} \mathbf{P} \otimes \mathbf{P}u_{j_1, i_1} \cdots u_{j_n, i_n} \mathbf{P} \right) \\ &= \sum_{l_1, \dots, l_k=1}^n X_{l_1} \cdots X_{l_k} \otimes (\Delta \mathbf{P}u_{l_1, i_1} \cdots u_{l_n, i_n} \mathbf{P}) \\ &= (id \otimes \Delta) \sum_{j_1, \dots, j_k=1}^n X_{l_1} \cdots X_{l_k} \otimes (\mathbf{P}u_{l_1, i_1} \cdots u_{l_n, i_n} \mathbf{P}) \\ &= (id \otimes \Delta)L_n(X_{i_1} \cdots X_{i_k}). \end{aligned}$$

We will call L_n the linear coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \dots, X_n \rangle$. The algebraic coaction will be defined in section 7.

Lemma 3.7. *Let L_n be the linear coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \dots, X_n \rangle$, $\{u_{i,j}\}_{i,j=1,\dots,n}$ and \mathbf{P} be the standard generators of $\mathcal{B}_s(n)$. Then,*

$$L_n(p_1(X_{i_1}) \cdots p_k(X_{i_k})) = \sum_{j_1, \dots, j_k=1}^n p_1(X_{j_1}) \cdots p_k(X_{j_k}) \otimes \mathbf{P} u_{j_1, i_1} \cdots u_{j_k, i_k} \mathbf{P},$$

for all $i_1 \neq i_2 \neq \cdots \neq i_k$ and $p_1, \dots, p_k \in \mathbb{C}\langle X \rangle$.

Proof. Since the map is linear, it suffices to show that the equation holds by assuming $p_l(X) = X^{t_l}$ where $t_l \geq 1$ for all $l = 1, \dots, k$. Then, we have

$$\begin{aligned} & L_n(\underbrace{x_{i_1} \cdots x_{i_1}}_{t_1 \text{ times}} \cdots \underbrace{x_{i_k} \cdots x_{i_k}}_{t_k \text{ times}}) \\ &= \sum_{j_{1,1}, \dots, j_{1,t_1}, \dots, j_{k,1}, \dots, j_{k,t_k}=1}^n x_{j_{1,1}} \cdots x_{j_{1,t_1}} \cdots x_{j_{k,1}} \cdots x_{j_{k,t_k}} \otimes \mathbf{P} u_{j_{1,1} i_1} \cdots u_{j_{k,t_k} i_k} \mathbf{P}. \end{aligned}$$

Notice that $u_{j_m, s} u_{j_m, s+1} = \delta_{j_m, s, j_m, s+1} u_{j_m, s}$, the right hand side of the above equation becomes

$$\sum_{j_1, \dots, j_k=1}^n x_{j_1}^{t_1} \cdots x_{j_k}^{t_k} \otimes \mathbf{P} u_{j_1 i_1} \cdots u_{j_k i_k} \mathbf{P}.$$

The proof is now completed \square

We will be using the following invariance condition to characterize conditionally boolean independence.

Definition 3.8. *Let (\mathcal{A}, ϕ) be a noncommutative probability space and $(x_i)_{i \in \mathbb{N}}$ be an infinite sequence of random variables in \mathcal{A} , we say the joint distribution satisfies the invariance conditions associated with the linear coactions of the boolean quantum permutation semigroups $\mathcal{B}_s(n)$ if for all n , we have*

$$\mu_{x_1, \dots, x_n}(p) \mathbf{P} = \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}(L_n p)$$

for all $p \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, where μ_{x_1, \dots, x_n} is the joint distribution of x_1, \dots, x_n .

Let $\{\bar{u}_{ij}\}_{i,j=1,\dots,n}$ be the standard generators of $\mathcal{A}_s(n)$, and $\{u_{ij}\}_{i,j=1,\dots,n} \cup \{\mathbf{P}\}$ be the standard generators of $B_s(n)$, then there exists a C^* -homomorphism $\beta : B_s(n) \rightarrow \mathcal{A}_s(n)$ such that:

$$\beta(u_{ij}) = \bar{u}_{ij}, \quad \beta(\mathbf{P}) = 1_{\mathcal{A}_s(n)}.$$

The C^* -homomorphism is well defined because of the universality of $B_s(n)$. Let $p = X_{i_1} \cdots X_{i_k} \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, then

$$\mu_{x_1, \dots, x_n}(p) \mathbf{P} = \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}(L_n p)$$

implies

$$\begin{aligned} & \mu_{x_1, \dots, x_n}(p) \mathbf{P} = \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}(L_n p) \\ & \mu_{x_1, \dots, x_n}(X_{i_1} \cdots X_{i_k}) \mathbf{P} = \sum_{j_1, \dots, j_k=1}^n (\mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)})(X_{j_1} \cdots X_{j_k} \otimes \mathbf{P} u_{j_1, i_1} \cdots u_{j_n, i_n} \mathbf{P}). \end{aligned}$$

Now, apply β on both sides of the above equation, we get

$$\mu_{x_1, \dots, x_n}(X_{i_1} \cdots X_{i_k}) 1_{\mathcal{A}_s(n)} = \sum_{j_1, \dots, j_k=1}^n (\mu_{x_1, \dots, x_n} \otimes id_{\mathcal{A}_s(n)})(X_{j_1} \cdots X_{j_k} \otimes \bar{u}_{j_1, i_1} \cdots \bar{u}_{j_n, i_n}),$$

which is the free quantum invariance condition. Since p is arbitrary, we have the following:

Proposition 3.9. *Let (\mathcal{A}, ϕ) be a noncommutative probability space and $(x_i)_{i=1, \dots, n}$ be a sequence of random variables in \mathcal{A} , the joint distribution of $(x_i)_{i=1, \dots, n}$ is invariant under the free quantum permutations $\mathcal{A}_s(n)$ if it satisfies the invariance condition associated with the linear coaction of the boolean quantum permutation semigroup $\mathcal{B}_s(n)$.*

4. BOOLEAN INDEPENDENCE AND FREENESS

In this section, we will show that operator valued boolean independent variables are sometimes operator valued free independent. Therefore, we should not be surprised that the joint distribution of any sequence of identically boolean independent random variables is invariant under the coaction of the free quantum permutations. Especially, in section 7, operator valued boolean independent variables are always operator valued free independent when we construct our conditional expectation in the unital-tail algebra case. The properties are related to the C^* -algebra unitalization. We provide a brief review here:

To every C^* algebra \mathcal{A} one can associate a unital C^* algebra $\bar{\mathcal{A}}$ which contains \mathcal{A} as a two-sided ideal and with the property that the quotient C^* -algebra $\bar{\mathcal{A}}/\mathcal{A}$ is isomorphic to \mathbb{C} . Actually, $\bar{\mathcal{A}} = \{x\bar{I} + a \mid x \in \mathbb{C}, a \in \mathcal{A}\}$, where \bar{I} is the unit of $\bar{\mathcal{A}}$. We will denote $x\bar{I} + a$ by (x, a) where $x \in \mathbb{C}$ and $a \in \mathcal{A}$, then we have

$$(x, a) + (y, b) = (x + y, a + b), \quad (x, a)(y, b) = (xy, ab + a + b), \quad (x, a)^* = (\bar{x}, a^*).$$

Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator-valued probability space where \mathcal{A} and \mathcal{B} are not necessarily unital. Let $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ be the unitalization defined above, then we can extend ρ to $\bar{\rho}$ s.t. $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$ is also an operator-valued probability space where \bar{E} is a conditional expectation on $\bar{\mathcal{A}}$.

It is natural to define \bar{E} as

$$\bar{E}[x, a] = (x, E[a]).$$

$\bar{E}[(1, 0)] = (1, 0)$, so \bar{E} is unital. The linear property is easy to check.

Take $(x_1, b_1), (x_2, b_2) \in \bar{\mathcal{B}}$ and $(y, a) \in \bar{\mathcal{A}}$, we have

$$\begin{aligned} \bar{E}[(x_1, b_1)(y, a)(x_2, b_2)] &= \bar{E}[x_1 y x_2, x_1 x_2 a + y x_2 b + x_2 b_1 a + x_1 b_2 + y b_1 b_2 + b_1 a b_2] \\ &= (x_1 y x_2, E[x_1 x_2 a + y x_2 b + x_2 b_1 a + x_1 b_2 + y b_1 b_2 + b_1 a b_2]) \\ &= (x_1 y x_2, x_1 x_2 E[a] + y x_2 b + x_2 b_1 E[a] + x_1 b_2 + y b_1 b_2 + b_1 E[a] b_2) \\ &= (x_1, b_1)(y, E[a])(x_2, b_2) \\ &= (x_1, b_1) \bar{E}[(y, a)](x_2, b_2). \end{aligned}$$

It is obvious that $\bar{E}^2 = \bar{E}$. Hence, \bar{E} is a $\bar{\mathcal{B}}\text{-}\bar{\mathcal{B}}$ bimodule from the unital algebra $\bar{\mathcal{A}}$ to the unital subalgebra $\bar{\mathcal{B}}$, i.e. a conditional expectation.

Proposition 4.1. *Let $(\mathcal{A}, \mathcal{B}, E) : \mathcal{A} \rightarrow \mathcal{B}$ be an operator valued probability space, $\{\mathcal{A}_i\}_{i \in I}$ be a \mathcal{B} -boolean independent family of sub-algebras and $\mathcal{B} \subset \mathcal{A}_i$ for all i . Then, in the unitalization operator probability space $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$, $\{\bar{\mathcal{A}}_i\}_{i \in I}$ is a $\bar{\mathcal{B}}$ -free independent family of sub-algebras.*

Proof. Let $(x, a) \in \bar{\mathcal{A}}$, where $a \in \mathcal{A}$ and x is a complex number, then $\bar{E}[(x, a)] = (x, E[a])$, thus $\bar{E}[(x, a)] = 0$ iff $x = 0$ and $E[a] = 0$.

Now, we can check the freeness directly. Let $(x_k, a_k) \in \bar{\mathcal{A}}_{i_k}$, i.e. $a_k \in \mathcal{A}_{i_k}$ and x_i 's are

complex numbers, for $k = 1, \dots, n$ and $\bar{E}[x_k, a_k] = 0$ and $i_1 \neq i_2 \neq \dots \neq i_n$, then we have $x_k = 0$ for all $k = 1, \dots, n$ and

$$\begin{aligned} \bar{E}[(x_1, a_1)(x_2, a_2) \cdots (x_n, a_n)] &= \bar{E}[(0, a_1)(0, a_2) \cdots (0, a_n)] \\ &= \bar{E}[(0, a_1 a_2 \cdots a_n)] \\ &= (0, E[a_1 a_2 \cdots a_n]) \\ &= (0, E[a_1]E[a_2] \cdots E[a_n]) \\ &= (0, 0) = 0. \end{aligned}$$

and $\bar{\mathcal{B}} \subset \bar{\mathcal{A}}_i$ for all i . \square

The examples for this proposition will be given in section 7.3. By checking the conditions for operator valued freeness directly as we did in the above theorem, we have

Corollary 4.2. *Let $(\mathcal{A}, \mathcal{B}, E) : \mathcal{A} \rightarrow \mathcal{B}$ be an operator valued probability space, $\{\mathcal{B}_i\}_{i \in I}$ be a \mathcal{B} -free independent family of sub-algebras. Then, in their unitalization operator probability space $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$, $\{\bar{\mathcal{A}}_i\}_{i \in I}$ is a $\bar{\mathcal{B}}$ -free independent family of sub-algebras.*

5. OPERATOR VALUED BOOLEAN RANDOM VARIABLES ARE INVARIANT UNDER BOOLEAN QUANTUM PERMUTATIONS

Let $\mathcal{B}_s(n)$ be the boolean permutation quantum semigroup of n with standard generators $\{u_{i,j}\}_{i,j=1,\dots,n}$ and \mathbf{P} . In this section, we prove that the joint distribution of n boolean independent operator valued random variables are invariant under the linear coactions of $\mathcal{B}_s(n)$. The following equality is the key to the proof of the statement: Fix k and $1 \leq i_1, \dots, i_k \leq n$, we have

$$\begin{aligned} &\sum_{j_1, \dots, j_k=1}^n \mathbf{P} u_{i_1, j_1} \cdots u_{i_k, j_k} \mathbf{P} \\ &= \sum_{j_1, \dots, j_{k-1}=1}^n \mathbf{P} u_{i_1, j_1} \cdots u_{i_{k-1}, j_{k-1}} \left(\sum_{j_k=1}^n u_{i_k, j_k} \mathbf{P} \right) \\ &= \sum_{j_1, \dots, j_{k-1}=1}^n \mathbf{P} u_{i_1, j_1} \cdots u_{i_{k-1}, j_{k-1}} \mathbf{P} \\ &= \cdots = \mathbf{P}. \end{aligned}$$

According to the definition of $\mathcal{B}_s(n)$, it follows that the product $u_{i_1, j_1} \cdots u_{i_k, j_k}$ is not vanishing only if it satisfies that $i_t \neq i_{t+1}$ whenever $j_t \neq j_{t+1}$ for all $1 \leq t \leq k-1$.

Given a set S , a collection of disjoint nonempty sets $P = \{V_i | i \in I\}$ is called a partition of S if $\bigcup_{i \in I} V_i = S$, $V_i \in P$ is called a block of the partition P . Let S be a finite ordered set, then all the partitions of S have finite blocks. A partition $P = \{V_1, \dots, V_r\}$ of S is interval if there are no two distinct blocks V_i and V_j and elements $a, c \in V_i$ and $b, d \in V_j$ s.t. $a < b < c$ or $b < c < d$. An interval partition $P = \{W_s | 1 \leq s \leq r\}$ is ordered if $a < b$ for all $a \in W_s, b \in W_t$ and $s < t$. We denote by $P_I(S)$ the collection of ordered interval partitions of S .

Let I be an index set, $[k] = \{1, \dots, k\}$ is an ordered set with the natural order. Let $I^k = I \times I \times \cdots \times I$ be the k -fold Cartesian product of the index set I . A sequence of indices $(i_m)_{m=1, \dots, k} \in I^k$ is said to be compatible with an ordered interval partition $P = \{W_1, \dots, W_r\} \in P_I([k])$ if $i_a = i_b$ whenever a, b are in the same block and $i_a \neq i_b$

whenever a, b are in two consecutive blocks, i.e. W_s and W_{s+1} for some $1 \leq s \leq r$. One should pay attention that $i_a = i_b$ is allowed for $a \in W_s$ and $b \in W_{s+2}$ for some $1 \leq s \leq r$

Now, we define an equivalent relation $\sim_{P_I([k])}$ on I^k : two sequences of indices

$$(i_m)_{m=1, \dots, k} \sim_{P_I([k])} (j_m)_{m=1, \dots, k}$$

if the two sequences are both compatible with an ordered interval partition $P \in P_I([k])$.

Let $\mathcal{J} = (i_m)_{m=1, \dots, k}$, $\mathcal{J}' = (j_m)_{m=1, \dots, k} \in \{1, \dots, n\}^k$, we denote $\mathbf{P}u_{i_1, j_1} u_{i_2, j_2} \cdots u_{i_k, j_k} \mathbf{P}$ by $U_{\mathcal{J}, \mathcal{J}'}$.

Lemma 5.1. *Fix $k \in \mathbb{N}$, let $\mathcal{B}_s(n)$ be the boolean permutation quantum semigroup with standard generators $\{u_{i,j}\}_{i,j=1, \dots, n}$ and \mathbf{P} . Let $\mathcal{J}_1 = (i_1, \dots, i_k)$, $\mathcal{J}_2 = (j_1, \dots, j_k) \in [n]^k$ be two sequences of indices. Then, the product $U_{\mathcal{J}_1, \mathcal{J}_2}$ is not vanishing if $\mathcal{J}_1 \sim_{P_I([k])} \mathcal{J}_2$*

Proof. Suppose \mathcal{J}_i is compatible with an ordered interval partition P_i for $i = 1, 2$. Let $P_1 = \{W_1, \dots, W_{r_1}\}$ and $P_2 = \{W'_1, \dots, W'_{r_2}\}$, then $P_1 \neq P_2$ implies that there exists a t such that $W_t \neq W'_t$ for some $1 \leq t \leq \min\{r_1, r_2\}$. Take the smallest t , then $W_s = W'_s$ whenever $s < t$ and $W_t \neq W'_t$. Then, these two intervals begin with the same number but end with different numbers, in other words, we have either $W_t \subsetneq W'_t$ or $W'_t \subsetneq W_t$. Without loss of generality, we assume $W_t \subsetneq W'_t$, then there is a number q s.t. $q \in W_t$ but $q+1 \notin W_t$ and $q, q+1 \in W'_t$. Now, we have $i_q \neq i_{q+1}$ and $j_q = j_{q+1}$, thus

$$U_{\mathcal{J}_1, \mathcal{J}_2} = \mathbf{P}u_{i_1, j_1} \cdots u_{i_q, j_q} u_{i_{q+1}, j_{q+1}} \cdots u_{i_k, j_k} \mathbf{P} = 0.$$

□

Lemma 5.2. *Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \rightarrow \mathcal{B})$ be an operator valued probability space. Let $(x_i)_{i=1, \dots, n}$ be a sequence of n random variables which are identically distributed and boolean independent with respect to E . Given two sequences of indices $\mathcal{J} = (i_q)_{q=1, \dots, k}$, $\mathcal{J}' = (j_q)_{q=1, \dots, k} \in [n]^k$ and $\mathcal{J} \sim_{P_I([n])} \mathcal{J}'$, then*

$$E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{k-1} x_{i_k}] = E[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}],$$

where $b_1, \dots, b_{k-1} \in \mathcal{B} \cup \{I_{\mathcal{A}}\}$.

Proof. Suppose that \mathcal{J} and \mathcal{J}' are compatible with an ordered interval partition $P = \{W_1, \dots, W_r\}$. Assume that $W_1 = \{1, \dots, k_1\}$, $W_2 = \{k_1 + 1, \dots, k_2\}, \dots, W_r = \{k_{r-1} + 1, \dots, k\}$, then $i_{k_t} \neq i_{k_t+1}$ and $j_{k_t} \neq j_{k_t+1}$ for $t = 1, \dots, r$. For convenience, we let $k_r = k$, $k_0 = 0$ and $b_k = I_{\mathcal{A}}$, we have

$$\begin{aligned} & E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{k-1} x_{i_k}] \\ &= E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{n-1} x_{i_k} b_k] \\ &= E\left[\prod_{s=1}^r \left(\prod_{t=n_{s-1}+1}^{n_s} x_{i_t} b_t\right)\right] \\ &= \prod_{s=1}^r E\left[\prod_{t=n_{s-1}+1}^{n_s} x_{i_t} b_t\right] \\ &= \prod_{s=1}^r E\left[\prod_{t=n_{s-1}+1}^{n_s} x_{j_t} b_t\right] \\ &= E\left[\prod_{s=1}^r \prod_{t=n_{s-1}+1}^{n_s} x_{j_t} b_t\right] \\ &= E[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}]. \end{aligned}$$

□

We will write \sim_{P_I} short for $\sim_{P_I([k])}$ when there is no confusion.

Theorem 5.3. *Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \rightarrow \mathcal{B})$ be an operator valued probability space, \mathcal{A} be unital and $\{x_i\}_{i=1, \dots, n}$ be a sequence of n random variables in \mathcal{A} which is identically distributed and boolean independent with respect to E . Let ϕ be a linear functional on \mathcal{B} and $\bar{\phi}$ is a linear functional on \mathcal{A} where $\bar{\phi}(\cdot) = \phi(E[\cdot])$. Then, the joint distribution of the sequence $\{x_i\}_{i=1, \dots, n}$ with respect to $\bar{\phi}$ is invariant under the linear coaction of the boolean permutation quantum semigroup $\mathcal{B}_s(n)$.*

Proof. Fix $k \in \mathbb{N}$, and indices $1 \leq i_1, \dots, i_k \leq n$, and $b_1, \dots, b_{k-1} \in \mathcal{B} \cup \{I_{\mathcal{A}}\}$, where $I_{\mathcal{A}}$ is the unit of \mathcal{A} , by the two lemmas above we have

$$\begin{aligned}
& \sum_{j_1, j_2, \dots, j_k=1}^n E[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}] \otimes \mathbf{P} u_{i_1, j_1} \cdots u_{i_k, j_k} \mathbf{P} \\
&= \sum_{\substack{j_1, j_2, \dots, j_k=1 \\ (j_s)_{s=1, \dots, k} \sim_{P_I} (i_t)_{t=1, \dots, k}}}^n E[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}] \otimes \mathbf{P} u_{i_1, j_1} \cdots u_{i_k, j_k} \mathbf{P} \\
&= \sum_{\substack{j_1, j_2, \dots, j_k=1 \\ (j_s)_{s=1, \dots, k} \sim_{P_I} (i_t)_{t=1, \dots, k}}}^n E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{k-1} x_{i_k}] \otimes \mathbf{P} u_{i_1, j_1} \cdots u_{i_k, j_k} \mathbf{P} \\
&= \sum_{j_1, j_2, \dots, j_n=1}^k E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{k-1} x_{i_k}] \otimes \mathbf{P} u_{i_1, j_1} \cdots u_{i_k, j_k} \mathbf{P} \\
&= E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{k-1} x_{i_k}] \otimes \mathbf{P}.
\end{aligned}$$

Let $b_1, \dots, b_{k-1} = 1_{\mathcal{A}}$ and let $\phi \otimes id_{\mathcal{B}_s(n)}$ act on the two sides of the above equation then we have

$$\begin{aligned}
& \bar{\phi}(x_{i_1} x_{i_2} \cdots x_{i_k}) \mathbf{P} \\
&= \bar{\phi}(x_{i_1} x_{i_2} \cdots x_{i_k}) \mathbf{P} \\
&= \sum_{j_1, j_2, \dots, j_k=1}^n \bar{\phi}(x_{j_1} x_{j_2} \cdots x_{j_n}) \mathbf{P} u_{i_1, j_1} \cdots u_{i_k, j_k} \mathbf{P},
\end{aligned}$$

which is our desired conclusion. \square

6. PROPERTIES OF TAIL ALGEBRA FOR BOOLEAN INDEPENDENCE

In order to study boolean exchangeable sequences of random variables, we need to choose a suitable kind of noncommutative probability spaces. It is pointed by Hasebe [10] that the W^* -probability with faithful normal states does not contain boolean independent random variables with Bernoulli law. Therefore, in our work, it is necessary to consider W^* probability spaces with more general states rather than faithful states:

Definition 6.1. *Let \mathcal{A} be a von Neumann algebra, a normal state ϕ on \mathcal{A} is said to be non-degenerated if $x = 0$ whenever $\phi(axb) = 0$ for all $a, b \in \mathcal{A}$.*

Remark 6.2. *By proposition 7.1.15 in [13], if ϕ is a non-degenerated normal state on \mathcal{A} then the GNS representation associated to ϕ is faithful. A faithful normal state on \mathcal{A} is faithful on all \mathcal{A} 's subalgebras but a non-degenerated normal state on \mathcal{A} may not be necessarily non-degenerated on \mathcal{A} 's subalgebras.*

Let (\mathcal{A}, ϕ) be a W^* -probability space with a non-degenerated normal state ϕ . Suppose \mathcal{A} is generated by an infinite sequence of random variables $\{x_i\}_{i \in \mathbb{N}}$, whose joint distribution is invariant under the linear coaction of the quantum semigroups $\mathcal{B}_s(n)$.

Let \mathcal{A}_0 be the non-unital algebra over \mathbb{C} generated by $\{x_i\}_{i \in \mathbb{N}}$. In this section, we assume that the unit $1_{\mathcal{A}}$ of \mathcal{A} is contained in the weak closure of \mathcal{A}_0 . We will denote the GNS construction associated to ϕ by (\mathcal{H}, ξ, π) , then there is a linear map $\hat{\cdot} : \mathcal{A}_0 \rightarrow \mathcal{H}$ such that $\hat{a} = \pi(a)\xi$ for all $a \in \mathcal{A}_0$. In the usual sense, the tail algebra \mathcal{A}_{tail} of $\{x_i\}_{i \in \mathbb{N}}$ is defined by:

$$\mathcal{A}_{tail} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\},$$

where $vN\{x_k | k \geq n\}$ is the von Neumann algebra generated by $\{x_k | k \geq n\}$. We will call \mathcal{A}_{tail} unital tail algebra in this paper. In this section, the range algebra we use is a "non-unital tail algebra" \mathcal{T} . The non-unital tail algebra \mathcal{T} of $\{x_i\}_{i \in \mathbb{N}}$ is given by the follows:

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\},$$

where $W^*\{x_k | k \geq n\}$ is the WOT closure of the non-unital algebra generated by $\{x_k | k \geq n\}$. If the unit of \mathcal{A} is contained in \mathcal{T} , then \mathcal{T} is also the unital tail-algebra of $\{x_i\}_{i \in \mathbb{N}}$. For convenience, we denote \mathcal{A}_n by the non-unital algebra generated by $\{x_k | k > n\}$. Now, we turn to define our T -linear map, the method comes from [15]. Because we are dealing with von Neumann algebras with non-degenerated normal states which are more general than the faithful states, it is necessary to provide a complete construction here. In [14], the normal conditional expectation K\"ostler constructed via the shift of the random variables requires the sequence only to be spreadable. But in our situation, the existence of the normal linear map relies on the invariance under the quantum semigroups $\mathcal{B}_s(n)$'s.

Lemma 6.3. *Let \mathcal{A} be a von Neumann algebra generated by an infinite sequence of selfadjoint random variables $(x_i)_{i \in \mathbb{N}}$, ϕ be a non-degenerated normal state on \mathcal{A} . If the sequence $(x_i)_{i \in \mathbb{N}}$ is exchangeable in (\mathcal{A}, ϕ) , then there is a C^* -isomorphism $\alpha : \mathcal{A}_0^{\|\cdot\|} \rightarrow \mathcal{A}_1^{\|\cdot\|}$ such that,*

$$\alpha(x_i) = x_{i+1},$$

for all $i \in \mathbb{N}$, where $\mathcal{A}_i^{\|\cdot\|}$ is the C^* -algebra generated by \mathcal{A}_i .

Proof. Let (\mathcal{H}, ξ, π) be the GNS construction associated to ϕ , it follows that $\{\hat{a} | a \in \mathcal{A}_0\}$ is dense in \mathcal{H} . For each $n \in \mathbb{N}$, denote by $A_{[n]}$ the non-unital algebra generated by $\{x_i | i \leq n\}$. Then $\bigcup_{n=1}^{\infty} \{\pi(a)\xi | a \in A_{[n]}\}$ is dense in \mathcal{H} . Given $y \in \bigcup_{n=1}^{\infty} A_{[n]}$, there exists $N \in \mathbb{N}$ such that $y \in A_{[N]}$. We can assume $y = p(x_1, \dots, x_N)$ for some $p \in \mathbb{C}\langle X_1, \dots, X_N \rangle_0$, then we have

$$\begin{aligned} \|\pi(p(x_1, \dots, x_N))\xi\|^2 &= \phi(\pi(p(x_1, \dots, x_N))^*(p(x_1, \dots, x_N))) \\ &= \phi(p(x_2, \dots, x_{N+1})^*(p(x_2, \dots, x_{N+1}))) \\ &= \|\pi(p(x_2, \dots, x_{N+1}))\xi\|^2 \end{aligned}$$

We can define an isometry U from \mathcal{H} to its subspace \mathcal{H}_1 which is generated by $\{\hat{a} | a \in A_1\}$ by the following formula:

$$U\pi(x_{i_1} \cdots x_{i_k})\xi = \pi(x_{i_1+1} \cdots x_{i_k+1})\xi,$$

for all $i_1, \dots, i_k \in \mathbb{N}$.

Since ϕ gives a faithful representation to \mathcal{A} , it gives a faithful representation to $\mathcal{A}_0^{\|\cdot\|}$. For all $y \in \mathcal{A}_1$, according to the faithfulness, we have

$$\|y\|^2 = \sup\left\{\frac{\langle y^* y \hat{a}, \hat{a} \rangle}{\langle \hat{a}, \hat{a} \rangle} \mid a \in \mathcal{A}_0, \hat{a} \neq 0\right\} = \sup\left\{\frac{\phi(a^* y^* y a)}{\phi(a^* a)} \mid a \in \mathcal{A}_0, \phi(a^* a) \neq 0\right\}.$$

Denote by $(\mathcal{H}', \xi', \pi')$ the GNS representation of \mathcal{A}_1 associated to ϕ . Indeed, \mathcal{H}' can be treated as \mathcal{H}_1 . Because the identity of \mathcal{A} is contained in the weak*-closure of the non unital algebra generated by $(x_i)_{i \in \mathbb{N}}$, by the Kaplansky density theorem, there exists a bounded sequence $\{y_i \mid \|y_i\| \leq 1\} \in \bigcup_{n=1}^{\infty} A_{[n]}$ such that y_i converges to $1_{\mathcal{A}}$ in WOT. Therefore, $\pi(y_i)\xi$ converges to ξ in norm. Again, by the exchangeability of $(x_i)_{i \in \mathbb{N}}$ and $U\pi(y_i)\xi \in \{\hat{b} \mid b \in \mathcal{A}_1\}$ for all i , we have

$$\|U\pi(y_i)\xi\| = \|\pi(y_i)\xi\| \leq 1$$

and

$$\langle U\pi(y_i)\xi, \xi \rangle = \langle \pi(y_i)\xi, \xi \rangle \rightarrow 1.$$

Therefore, $U\pi(y_i)\xi$ converges to ξ in norm, namely, $\xi \in \mathcal{H}_1$.

Let $x \in \mathcal{A}_1$, then $x = p(x_2, \dots, x_{N+1})$ for some N and $p \in \mathbb{C}\langle X_1, \dots, X_N \rangle_0$. For every $y \in \mathcal{A}_0$ there exists an M , such that $y = p'(x_1, \dots, x_M)$ for some $p' \in \mathbb{C}\langle X_1, \dots, X_M \rangle_0$. By the exchangeability, we send x_1 to x_{N+M} . Then

$$\begin{aligned} \|\pi(x)\hat{y}\|_{\mathcal{H}}^2 &= \phi(p'(x_1, \dots, x_M)^* p(x_2, \dots, x_{N+1})^* p(x_2, \dots, x_{N+1}) p'(x_1, \dots, x_M)) \\ &= \phi(p'(x_{M+N}, \dots, x_M)^* p(x_2, \dots, x_{N+1})^* p(x_2, \dots, x_{N+1}) p'(x_{N+M}, x_2, x_3, \dots, x_M)) \\ &= \|\pi'(x)p'(x_{M+N}, x_2, \dots, x_M)\|_{\mathcal{H}'}^2 \end{aligned}$$

and

$$\|\widehat{p'(x_1, \dots, x_M)}\|_{\mathcal{H}} = \|\widehat{p'(x_{M+N}, x_2, \dots, x_M)}\|_{\mathcal{H}'}.$$

Therefore, we get

$$\left\{\frac{\|\pi(x)\hat{a}\|_{\mathcal{H}}}{\|\hat{a}\|_{\mathcal{H}}} \mid a \in \mathcal{A}_0, \hat{a} \neq 0\right\} \subseteq \left\{\frac{\|\pi'(x)\hat{a}\|_{\mathcal{H}'}}{\|\hat{a}\|_{\mathcal{H}'}} \mid a \in \mathcal{A}_1, \hat{a} \neq 0\right\},$$

which implies

$$\|x\| = \|\pi(x)\| = \sup\left\{\frac{\|\pi x \hat{a}\|_{\mathcal{H}}}{\|\hat{a}\|_{\mathcal{H}}} \mid a \in \mathcal{A}_0, \hat{a} \neq 0\right\} \leq \sup\left\{\frac{\|\pi'(x)\hat{a}\|_{\mathcal{H}'}}{\|\hat{a}\|_{\mathcal{H}'}} \mid a \in \mathcal{A}_1, \hat{a} \neq 0\right\} = \|\pi'(x)\|.$$

It follows that $\|x\| = \|\pi'(x)\|$ for all $x \in \mathcal{A}_1$. By taking the norm limit, we have $\|x\| = \|\pi'(x)\|$ for all $x \in \mathcal{A}_1^{\|\cdot\|}$, so the GNS representation of $\mathcal{A}_1^{\|\cdot\|}$ associated to ϕ is faithful.

Now, we turn to define our C^* -isomorphism α :

Since U is an isometric isomorphism from \mathcal{H} to \mathcal{H}' , we define a homomorphism $\alpha' : \pi(\mathcal{A}_0) \rightarrow B(\mathcal{H}')$ by the following formula

$$\alpha'(y) = UyU^*,$$

for $y \in \pi(\mathcal{A}_0)$. Let $y \in \pi(\mathcal{A}_{[n]})$, then $y = \pi(p(x_1, \dots, x_n))$ for some $p \in \mathbb{C}\langle X_1, \dots, X_n \rangle_0$.

For all $v \in \bigcup_{n=2}^{\infty} \{\pi(a)\xi \mid a \in A_{[n]} \subset \mathcal{H}'\}$, there exists $N \in \mathbb{N}$ and $p_1 \in \mathbb{C}\langle X_1, \dots, X_N \rangle_0$ such

that $v = \pi(p_1(x_2, \dots, x_{N+1}))\xi$. We have

$$\begin{aligned}\alpha'(y)v &= U\pi(p(x_1, \dots, x_n)U^*\pi(p_1(x_2, \dots, x_{N+1})))\xi \\ &= U\pi(p(x_1, \dots, x_n)\pi(p_1(x_1, \dots, x_N)))\xi \\ &= U\pi(p(x_1, \dots, x_n)p_1(x_1, \dots, x_N))\xi \\ &= \pi(p(x_2, \dots, x_{n+1})p_1(x_1, \dots, x_{N+1}))\xi\end{aligned}$$

Since $\bigcup_{n=2}^{\infty} \{\pi(a)\xi | a \in A_{[n]}\}$ is dense in \mathcal{H}_1 , we get $\alpha'(\pi(p(x_1, \dots, x_n))) = \pi(p(x_2, \dots, x_{n+1}))$.

Because (\mathcal{H}, ξ, π) and $(\mathcal{H}', \xi', \pi')$ are faithful GNS representations for \mathcal{A}_0 and \mathcal{A}_1 respectively, there is a well defined norm preserving homomorphism $\alpha : \mathcal{A}_0 \rightarrow \mathcal{A}_1$, such that $\alpha(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$. Therefore, α extends to a C^* -isomorphism from $\mathcal{A}_0^{\|\cdot\|}$ to $\mathcal{A}_1^{\|\cdot\|}$. \square

Since $W^*\{x_k | k \geq n\}$'s are WOT closed, their intersection is a WOT closed subset of \mathcal{A} . Following the proof of proposition 4.2 in [15], we have

Lemma 6.4. *For each $a \in \mathcal{A}_0$, $\{\alpha^n(a)\}_{n \in \mathbb{N}}$ is a bounded WOT convergent sequence. Therefore, there exists a well defined ϕ -preserving linear map $E : \mathcal{A}_0 \rightarrow \mathcal{T}$ by the following formula:*

$$E[a] = w^* - \lim_{n \rightarrow \infty} \alpha^n(a)$$

for $a \in \mathcal{A}_0$

Proof. By lemma 6.3, there is a norm preserving endomorphism α of \mathcal{A}_0 such that

$$\phi \circ \alpha = \phi \quad \text{and} \quad \alpha(x_i) = x_{i+1}.$$

For $I \subset \mathbb{N}$, denote by \mathcal{A}_I the non-unital algebra generated by $\{x_i | i \in I\}$. Suppose $a, b, c \in \bigcup_{|I| < \infty} \mathcal{A}_I$, we can assume $a \in \mathcal{A}_I, b \in \mathcal{A}_J$ and $c \in \mathcal{A}_K$ for some finite sets

$I, J, K \subset \mathbb{N}$. Because I, J, K are finite, there exists an N such that $(I \cup K) \cap (J + n) = \emptyset$, for all $n > N$. We infer from the exchangeability that $\phi(a\alpha^n(b)c) = \phi(a\alpha^{n+1}(b)c)$ for all $n > N$. This establishes the limit

$$\lim_{n \rightarrow \infty} \phi(a\alpha^n(b)c)$$

on the weak*-dense algebra $\bigcup_{|I| < \infty} \mathcal{A}_I$. We conclude from this and $\{\alpha^n(b)\}_{n \in \mathbb{N}}$ is bounded

that the pointwise limit of the sequence α defines a linear map $E : \mathcal{A}_0 \rightarrow \mathcal{A}$ such that $E(\mathcal{A}_0) \subset \mathcal{T}$. \square

To extend E to the W^* -algebra \mathcal{A} , we need to make use of the boolean invariance conditions.

Lemma 6.5. *Let (\mathcal{A}, ϕ) be a noncommutative probability space, $\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$ be an infinite sequence of random variables whose joint distribution is invariant under the linear coactions of the quantum semigroups $\mathcal{B}_s(k)$'s, then*

$$\phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, and $k_1, \dots, k_n \in \mathbb{N}$

Proof. If $i_l \neq i_m$ for all $l \neq m$, then the statement holds by the exchangeability of the sequence. Suppose the number i_l appears m times in the sequence, which are $\{i_{l_j}\}_{j=1, \dots, m}$ such that $i_{l_j} = i_l$ and $l_1 < l_2 < \dots < l_m$. Since the sequence is finite, with out losing generality, we can assume that $i_1, \dots, i_n \leq N+1$ and $i_{l_j} = N+1$ for some N by the exchangeability.

For each $M \in \mathbb{N}$, by lemma 4.2, we have the following representation π_M of the quantum semigroup $\mathcal{B}_s(M+N)$:

$$\pi_M(u_{i,j}) = \begin{cases} P_{i-N,j-N}, & \text{if } \min\{i,j\} > N \\ \delta_{i,j}P, & \text{if } \min\{i,j\} \leq N \end{cases},$$

and $\pi(\mathbf{P}) = P$, where $p_{i,j}$ and p are projections in $B(\mathbb{C}^{2M})$ given by lemma 4.2. Then we have

$$PP_{i,j}P = \frac{1}{M}P,$$

for $1 \leq i, j \leq N$.

According to the boolean invariance condition, we have:

$$\begin{aligned} & \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \dots x_{i_n}^{k_n})P \\ = & \sum_{j_1, j_2, \dots, j_n=1}^{M+N} \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \dots x_{i_n}^{k_n})P u_{j_1, i_1} \dots u_{j_n, i_n} P \\ = & \sum_{j_{l_1}, j_{l_2}, \dots, j_{l_m}=1}^N \phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) P P_{j_{l_1}, i_{l_1}} P P_{j_{l_2}, i_{l_2}} P \dots u_{j_{l_m}, i_{l_m}} P \\ = & \frac{1}{M^m} \sum_{j_{l_1}, j_{l_2}, \dots, j_{l_m}=1}^N \phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) P \\ = & \frac{1}{M^m} \left[\sum_{\substack{j_{l_s} \neq j_{l_t} \\ \text{if } s \neq t}}^N \phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) P + \sum_{j_{l_s} = j_{l_t} \text{ for some } s \neq t}^N \phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) P \right]. \end{aligned}$$

In the first part of the sum, by the exchangeability, it follows that

$$\phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1} \dots x_{N+1}^{k_{l_1}} \dots x_{N+2}^{k_{l_2}} \dots x_{i_n}^{k_n}),$$

where we sent j_{l_s} to $N+s$. Then, we have

$$\frac{1}{M^m} \sum_{\substack{j_{l_s} \neq j_{l_t} \\ \text{if } s \neq t}}^N \phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) P = \frac{\prod_{s=0}^{m-1} (M-s)}{M^m} \phi(x_{i_1}^{k_1} \dots x_{N+1}^{k_{l_1}} \dots x_{N+2}^{k_{l_2}} \dots x_{i_n}^{k_n}) P,$$

which converges to $\phi(x_{i_1}^{k_1} \dots x_{N+1}^{k_{l_1}} \dots x_{N+2}^{k_{l_2}} \dots x_{i_n}^{k_n}) P$ as M goes to ∞ .

To the second part of the sum, we have

$$\phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) \leq \|x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}\| \leq \|x_1^{k_1 + \dots + k_n}\|,$$

which is bounded, therefore,

$$\left| \frac{1}{M^m} \sum_{\substack{j_{l_s} = j_{l_t} \\ \text{for some } s \neq t}}^N \phi(x_{i_1}^{k_1} \dots x_{j_{l_1}}^{k_{l_1}} \dots x_{j_{l_2}}^{k_{l_2}} \dots x_{i_n}^{k_n}) \right| \leq \left(1 - \frac{\prod_{s=0}^{m-1} (M-s)}{M^m}\right) \|x_1^{k_1 + \dots + k_n}\|$$

goes to 0 as M goes to ∞ . By now, we have showed that if there are indices $i_s = i_t$ for $s \neq t$ in the the sequence, we can, with out changing the value of the mixed moments, change them to two different large numbers j_s, j_t such that j_s, j_t differ the other indices. After a finite steps, we will have

$$\phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_n}^{k_n}),$$

such that all the j_l ' are not equal to any of the other indices. By the exchangeability, the proof is complete. \square

Corollary 6.6. *Let $\{x_i\}_{i \in \mathbb{N}} \subset (\mathcal{A}, \phi)$ be an infinite sequence of random variables whose joint distribution is invariant under the linear coactions of the quantum semigroups $\mathcal{B}_s(k)$'s, then*

$$\phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_n}^{k_n}),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n, j_1 \neq j_2 \neq \cdots \neq j_n, k_1, \dots, k_n, j_1, \dots, j_n \in \mathbb{N}$. Moreover, we have

$$\phi(ax_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n} b) = \phi(ax_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_n}^{k_n} b),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n, j_1 \neq j_2 \neq \cdots \neq j_n, k_1, \dots, k_n, j_1, \dots, j_n > M$ and $a, b \in \mathcal{A}_{[M]}$ for some $M \in \mathbb{N}$.

Lemma 6.7. *For all $a, b, y \in \mathcal{A}_0$, we have*

$$\langle E(y)\hat{a}, \hat{b} \rangle = \langle y\widehat{E(a)}, \widehat{E[b]} \rangle.$$

Proof. Because an element in \mathcal{A}_0 is a finite linear combination of the noncommutative monomials, it suffices to show the property in the case: $b^* = x_{i_1}^{r_1} \cdots x_{i_l}^{r_l}, y = x_{j_1}^{s_1} \cdots x_{j_m}^{s_m}, a = x_{k_1}^{t_1} \cdots x_{k_n}^{t_n}$, where $i_1 \neq i_2 \neq \cdots \neq i_l, j_1 \neq \dots \neq j_m, k_1 \neq \dots \neq k_n$ and all the power indices are positive integers. Let $N = \max\{i_1, \dots, i_l, j_1, \dots, j_m, k_1, \dots, k_n\}$, for all $L > N$, we have $i_l \neq j_1 + L$ and $j_m + L \neq k_1$. Therefore, we have

$$\begin{aligned} \langle E(y)\hat{a}, \hat{b} \rangle &= \lim_{M \rightarrow \infty} \langle \alpha^M(y)\hat{a}, \hat{b} \rangle \\ &= \langle \alpha^L(y)\hat{a}, \hat{b} \rangle \\ &= \phi(x_{i_1}^{r_1} \cdots x_{i_l}^{r_l} x_{j_1+L}^{s_1} \cdots x_{j_m+L}^{s_m} x_{k_1}^{t_1} \cdots x_{k_n}^{t_n}), \end{aligned}$$

by corollary 6.6,

$$\begin{aligned} &= \phi(x_1^{r_1} \cdots x_l^{r_l} x_{l+1}^{s_1} \cdots x_{l+m}^{s_m} x_{l+m+1}^{t_1} \cdots x_{l+m+n}^{t_n}) \\ &= \phi(x_1^{r_1} \cdots x_l^{r_l} x_{l+1}^{s_1} \cdots x_{l+m}^{s_m} x_{l+m+1}^{t_1} \cdots x_{l+m+n}^{t_n}) \\ &= \phi(x_{i_1+L}^{r_1} \cdots x_{i_l+L}^{r_l} x_{j_1}^{s_1} \cdots x_{j_m}^{s_m} x_{k_1+2L}^{t_1} \cdots x_{k_n+2L}^{t_n}) \\ &= \phi(\alpha^L(x_{i_1}^{r_1} \cdots x_{i_l}^{r_l}) x_{j_1}^{s_1} \cdots x_{j_m}^{s_m} \alpha^{2L}(x_{k_1}^{t_1} \cdots x_{k_n}^{t_n})) \\ &= \lim_{M \rightarrow \infty} \phi(\alpha^N(b^*) y \alpha^{2L+M}(a)) \\ &= \phi(\alpha^L(b^*) y E[a]). \end{aligned}$$

Notice that $\{\alpha^L(b)|L \leq N\}$ is a bounded sequence of random variables which converges to $E[b^*]$ in WOT and $\phi(\cdot y E[a])$ is a normal linear functional on \mathcal{A} , we have

$$\begin{aligned} \phi(\alpha^L(b^*) y E[a]) &= \lim_{M \rightarrow \infty} \phi(\alpha^M(b^*) y E[a]) \\ &= \phi(E[b]^* y E[a]) \\ &= \langle y \widehat{E[a]}, \widehat{E[b]} \rangle. \end{aligned}$$

\square

Lemma 6.8. *Let $\{y_n\}_{n \in \mathbb{N}} \subset \mathcal{A}_0$ be a bounded sequence of random variables such that $w^* - \lim y_n = 0$, then $w^* - \lim E[y_n] = 0$.*

Proof. For all $a, b \in \mathcal{A}_0$, we have

$$\lim_n \langle E[y_n] \hat{a}, \widehat{E[b]} \rangle = \lim_n \langle y_n \widehat{E[a]}, \widehat{E[b]} \rangle = 0.$$

Since $\{\hat{a} | a \in \mathcal{A}_0\}$ is dense in \mathcal{H}_ξ , we get our desired conclusion. \square

Let $y \in \mathcal{A}$ and $\{y_n\}_{n \in \mathbb{N}} \subset \mathcal{A}_0$ be a bounded sequence such that y_n converges to y in WOT. For all $a, b \in \mathcal{A}_0$, we have

$$\lim_n \langle E[y_n] \hat{a}, \hat{b} \rangle = \lim_n \langle y_n \widehat{E[a]}, \widehat{E[b]} \rangle = \langle y \widehat{E[a]}, \widehat{E[b]} \rangle.$$

Therefore, $\{E[y_n]\}_{n \in \mathbb{N}}$ converges to an element y' in pointwise weak topology, by the lemma above, we see that y' is independent of the choice of $\{y_n\}_{n \in \mathbb{N}}$. Since $\{E[y_n]\}_{n \in \mathbb{N}} \subset \mathcal{T}$, we have $y' \in \mathcal{T}$. By now, we have defined a linear map $E : \mathcal{A} \rightarrow \mathcal{T}$ and we have

Lemma 6.9. *E is normal.*

Proof. Let $\{y_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ be a bounded WOT convergent sequence of random variables such that $w^* - \lim_{n \rightarrow \infty} y_n = y$. Then, we have

$$\lim_{n \rightarrow \infty} \langle E[y_n] \hat{a}, \hat{b} \rangle = \lim_{n \rightarrow \infty} \langle y_n \widehat{E[a]}, \widehat{E[b]} \rangle = \langle y \widehat{E[a]}, \widehat{E[b]} \rangle = \langle E[y] \hat{a}, \hat{b} \rangle,$$

for all $a, b \in \mathcal{A}_0$. Therefore, E is normal. \square

Now, we can turn to show that E is a conditional expectation from \mathcal{A} to ϕ :

Lemma 6.10. *$E[a] = a$ for all $a \in \mathcal{T}$.*

Proof. Let $a \in \mathcal{T}$, $b, c \in \mathcal{A}_0$, then there exists an $N \in \mathbb{N}$ such that $a \in \overline{\mathcal{A}_{N+1}}^{w^*}$ and $b, c \in \mathcal{A}_{[N]}$. We can approximate a in WOT by a bounded sequence $(a_k)_{k \in \mathbb{N}} \subset \mathcal{A}_{N+1}$ in WOT. According to the definition of E and the exchangeability, we have

$$\begin{aligned} \langle E[a] \hat{c}, \hat{b} \rangle &= \phi(b^* E[a] c) \\ &= \lim_k \phi(b^* E[a_k] c) \\ &= \lim_k \lim_n \phi(b^* \alpha^n(a_k) c) \\ &= \lim_k \phi(b^* a_k c) \\ &= \phi(b^* a c) = \langle a \hat{c}, \hat{b} \rangle. \end{aligned}$$

The equation is true for all $b, c \in \mathcal{A}_0$, so $E[a] = a$. \square

To check the bimodule property of E , we need to show that the quality of 6.7 holds for all $x \in \mathcal{A}$:

Lemma 6.11. *For all $a, b, x \in \mathcal{A}$, we have*

$$\phi(a E[x] b) = \phi(E[a] x E[b]).$$

Proof. By the Kaplansky's density theorem, there exist two bounded sequences $\{a_n \in \mathcal{A}_0 \mid \|a_n\| \leq \|a\|, n \in \mathbb{N}\}$ and $\{b_n \in \mathcal{A}_0 \mid \|b_n\| \leq \|b\|, n \in \mathbb{N}\}$ which converge to a and b in WOT, respectively. Since ϕ and E are normal, we have

$$\begin{aligned}
\phi(aE[x]b) &= \lim_n \phi(a_n E[x]b) \\
&= \lim_n \lim_m \phi(a_n E[x]b_m) \\
&= \lim_n \lim_m \phi(E[a_n]xE[b_m]) \\
&= \lim_n \phi(E[a_n]xE[b]) \\
&= \phi(E[a]xE[b]).
\end{aligned}$$

□

Lemma 6.12. $E[ax] = aE[x]$ for all $a \in \mathcal{T}$ and $x \in \mathcal{A}$.

Proof. For all $b, c \in \mathcal{A}_0$, by lemma 6.11 and Lemma 6.10, we have

$$\begin{aligned}
\langle E[ax]\hat{b}, \hat{c} \rangle &= \phi(c^* E[ax]b) \\
&= \phi(E[c^*]axE[b]) \\
&= \phi((E[c^*]a)xE[b]).
\end{aligned}$$

since $E[c^*]a \in \mathcal{T}$, $E[E[c^*]a] = E[c^*]a$, then

$$\begin{aligned}
\phi((E[c^*]a)xE[b]) &= \phi(E[E[c^*]a]xE[b]) \\
&= \phi(E[c^*]aE[x]b) \\
&= \phi(E[c^*]E[aE[x]]b) \\
&= \phi(E[E[c^*]](aE[x])E[b]) \\
&= \phi(E[c^*](aE[x])E[b]) \\
&= \phi(c^* E[aE[x]]b) \\
&= \phi(c^* aE[x]b) \\
&= \langle aE[x]\hat{b}, \hat{c} \rangle.
\end{aligned}$$

Since b, c are arbitrary, we get our desired conclusion

□

Lemma 6.13.

$$E[x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n}] = E[x_{i_1}^{k_1} \cdots \alpha^N(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}) \cdots x_{i_n}^{k_n}]$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $N \geq \max\{i_1, \dots, i_n\}$, k_j 's are positive integers.

Proof. Given $a, b \in \mathcal{A}_0$, then there exists an M such that $a, b \in \mathcal{A}_{[M]}$. Then, we have

$$\begin{aligned}
&\langle E[x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n}]\hat{a}, \hat{b} \rangle \\
&= \lim_{l \rightarrow \infty} \langle \alpha^l(x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n})\hat{a}, \hat{b} \rangle \\
&= \langle \alpha^M(x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n})\hat{a}, \hat{b} \rangle \\
&= \langle x_{i_1+M}^{k_1} \cdots x_{i_s+M}^{k_s} \cdots x_{i_t+M}^{k_t} \cdots x_{i_n+M}^{k_n}\hat{a}, \hat{b} \rangle,
\end{aligned}$$

by lemma 6.6 and $i_1 + M \neq \dots \neq i_{s-1} + M \neq i_s + M + N \neq i_{s+1} + M + N \neq \dots \neq i_t + M + N \neq i_{t+1} + M \neq \dots i_n + M$,

$$\begin{aligned} & \langle x_{i_1+M}^{k_1} \dots x_{i_s+M}^{k_s} \dots x_{i_t+M}^{k_t} \dots x_{i_n+M}^{k_n} \hat{a}, \hat{b} \rangle \\ &= \langle x_{i_1+M}^{k_1} \dots x_{i_s+M+N}^{k_s} \dots x_{i_t+M+N}^{k_t} \dots x_{i_n+M}^{k_n} \hat{a}, \hat{b} \rangle \\ &= \langle \alpha^M(x_{i_1}^{k_1} \dots \alpha^N(x_{i_s}^{k_s} \dots x_{i_t}^{k_t}) \dots x_{i_n}^{k_n}) \hat{a}, \hat{b} \rangle \\ &= \lim_{l \rightarrow \infty} \langle \alpha^l(x_{i_1}^{k_1} \dots \alpha^N(x_{i_s}^{k_s} \dots x_{i_t}^{k_t}) \dots x_{i_n}^{k_n}) \hat{a}, \hat{b} \rangle \\ &= \langle E[x_{i_1}^{k_1} \dots \alpha^N(x_{i_s}^{k_s} \dots x_{i_t}^{k_t}) \dots x_{i_n}^{k_n}] \hat{a}, \hat{b} \rangle. \end{aligned}$$

Because $\{\hat{a} | a \in \mathcal{A}_0\}$ is dense in \mathcal{H} , the proof is complete. \square

Corollary 6.14.

$$E[x_{i_1}^{k_1} \dots x_{i_s}^{k_s} \dots x_{i_t}^{k_t} \dots x_{i_n}^{k_n}] = E[x_{i_1}^{k_1} \dots E[x_{i_s}^{k_s} \dots x_{i_t}^{k_t}] \dots x_{i_n}^{k_n}],$$

whenever $i_1 \neq i_2 \neq \dots \neq i_n$.

Proof. Let $N = \max\{i_1, \dots, i_n\}$. Since $E[x_{i_s}^{k_s} \dots x_{i_t}^{k_t}] = w^* - \lim_{l \rightarrow \infty} \alpha^l(x_{i_s}^{k_s} \dots x_{i_t}^{k_t})$, we have

$$E[x_{i_s}^{k_s} \dots x_{i_t}^{k_t}] = w^* - \lim_{l \rightarrow \infty} \frac{1}{l} \sum_{s=1}^l \alpha^{N+l}(x_{i_s}^{k_s} \dots x_{i_t}^{k_t}).$$

Then, by lemma 6.13,

$$\begin{aligned} & E[x_{i_1}^{k_1} \dots x_{i_s}^{k_s} \dots x_{i_t}^{k_t} \dots x_{i_n}^{k_n}] \\ &= \frac{1}{l} \sum_{s=1}^l E[x_{i_1}^{k_1} \dots \alpha^{N+l}(x_{i_s}^{k_s} \dots x_{i_t}^{k_t}) \dots x_{i_n}^{k_n}] \\ &= E[x_{i_1}^{k_1} \dots [w^* - \lim_{l \rightarrow \infty} \frac{1}{l} \sum_{s=1}^l \alpha^{N+l}(x_{i_s}^{k_s} \dots x_{i_t}^{k_t})] \dots x_{i_n}^{k_n}] \\ &= E[x_{i_1}^{k_1} \dots E[x_{i_s}^{k_s} \dots x_{i_t}^{k_t}] \dots x_{i_n}^{k_n}]. \end{aligned}$$

The last two equations follow the normality of E and

$$x_{i_1}^{k_1} \dots \left[\frac{1}{l} \sum_{s=1}^l \alpha^{N+l}(x_{i_s}^{k_s} \dots x_{i_t}^{k_t}) \right] \dots x_{i_n}^{k_n} \rightarrow x_{i_1}^{k_1} \dots E[x_{i_s}^{k_s} \dots x_{i_t}^{k_t}] \dots x_{i_n}^{k_n}$$

in WOT. \square

Lemma 6.15.

$$E[b_1 x_{i_1}^{k_1} b_2 \dots b_s x_{i_s}^{k_s} \dots b_t x_{i_t}^{k_t} \dots b_n x_{i_n}^{k_n}] = E[b_1 x_{i_1}^{k_1} b_2 \dots E[b_s x_{i_s}^{k_s} \dots b_t x_{i_t}^{k_t}] \dots b_n x_{i_n}^{k_n}],$$

whenever $i_1 \neq i_2 \neq \dots \neq i_n$, k_1, \dots, k_n are positive integers, $b_1, \dots, b_n \in \mathcal{A}_{N+1}$ where $N = \max\{i_1, \dots, i_n\}$.

Proof. By the linearity of E , we can assume that b_i 's are "monomials", i.e. $b_j = x_{i_{j,1}} \dots x_{i_{j,r_j}}$ where $i_{j,j'}$'s are greater than N . Then,

$$b_1 x_{i_1}^{k_1} b_2 \dots b_s x_{i_s}^{k_s} \dots b_t x_{i_t}^{k_t} \dots b_n x_{i_n}^{k_n} = b_1 x_{i_1}^{k_1} b_2 \dots x_{i_{s,1}} \dots x_{i_{s,r_s}} x_{i_s}^{k_s} \dots x_{i_{t,1}} \dots x_{i_{t,r_t}} x_{i_t}^{k_t} \dots b_n x_{i_n}^{k_n},$$

$i_{s,1} \geq N+1 > i_{s-1}$ and $i_{t,r_t} \geq N+1 > i_{t+1}$. Therefore, by lemma 6.14,

$$\begin{aligned} & E[b_1 x_{i_1}^{k_1} b_2 \dots x_{i_{s,1}} \dots x_{i_{s,r_s}} x_{i_s}^{k_s} \dots x_{i_{t,1}} \dots x_{i_{t,r_t}} x_{i_t}^{k_t} \dots b_n x_{i_n}^{k_n}] \\ &= E[b_1 x_{i_1}^{k_1} b_2 \dots E[x_{i_{s,1}} \dots x_{i_{s,r_s}} x_{i_s}^{k_s} \dots x_{i_{t,1}} \dots x_{i_{t,r_t}} x_{i_t}^{k_t}] \dots b_n x_{i_n}^{k_n}] \\ &= E[b_1 x_{i_1}^{k_1} b_2 \dots E[b_s x_{i_s}^{k_s} \dots b_t x_{i_t}^{k_t}] \dots b_n x_{i_n}^{k_n}]. \end{aligned}$$

□

Proposition 6.16. *Let (\mathcal{A}, ϕ) be a W^* -probability space and $(x_i)_{i \in \mathbb{N}}$ be a sequence of selfadjoint random variables in \mathcal{A} whose joint distribution is invariant of under the boolean permutations. Let E be the conditional expectation onto the non-unital tail algebra \mathcal{T} of the sequence. Then, E has the following factorization property: for all $n, k \in \mathbb{N}$, polynomials $p_1, \dots, p_n \in \mathcal{T}\langle X_1, \dots, X_k \rangle_0$ and $i_1, \dots, i_n \in \{1, \dots, k\}$, we have*

$$E[p_1(x_{i_1}) \cdots p_l(x_{i_m}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1}) \cdots E[p_l(x_{i_m})] \cdots p_n(x_{i_n})].$$

Proof. It suffices to prove the statement in the case: p_1, \dots, p_n are \mathcal{T} -monomials but none of them is an element of \mathcal{T} . Assume that

$$p_i(X) = b_{i,0} X^{t_{i,1}} b_{i,1} X^{t_{i,2}} b_{i,2} \cdots X_{t_i}^{k_i},$$

where $b_{i,j} \in \mathcal{T}$ and $t'_{i,j}$ s are positive integers. Let $N = \max\{i_1, \dots, i_n\}$, then $b_{i,j} \in \mathcal{T} \subset \overline{\mathcal{A}_{N+1}}^{w^*}$. By the Kaplansky theorem, for every $b_{i,j}$, there exists a bounded sequence $\{b_{l,i,j}\}_{l \in \mathbb{N}}$ such that $b_{l,i,j}$ converges to $b_{i,j}$ in strong operator topology (SOT). Let $p_{n,i}(X) = b_{n,i,0} X^{t_{i,1}} b_{n,i,1} X^{t_{i,2}} b_{n,i,2} \cdots X_{t_i}^{k_i}$, then $p_{l,k}(x_{i_k})$ converges to $p_k(x_{i_k})$ in SOT. By the normality of E , we have

$$E[p_1(x_{i_1}) \cdots p_l(x_{i_m}) \cdots p_n(x_{i_n})] = w^* - \lim_{l \rightarrow \infty} E[p_{l,1}(x_{i_1}) \cdots p_{l,m}(x_{i_m}) \cdots p_{l,n}(x_{i_n})].$$

By lemma 6.15, we have

$$E[p_{l,1}(x_{i_1}) \cdots p_{l,m}(x_{i_m}) \cdots p_{l,n}(x_{i_n})] = E[p_{l,1}(x_{i_1}) \cdots E[p_{l,m}(x_{i_m})] \cdots p_{l,n}(x_{i_n})].$$

It follows that $E[p_{l,m}(x_{i_m})]$ converges to $E[p_m(x_{i_m})]$ in WOT. Therefore, $p_{l,1}(x_{i_1}) \cdots E[p_{l,m}(x_{i_m})] \cdots p_{l,n}(x_{i_n})$ converges to $p_1(x_{i_1}) \cdots E[p_m(x_{i_m})] \cdots p_n(x_{i_n})$ in WOT. Now, we have

$$\begin{aligned} & E[p_1(x_{i_1}) \cdots p_l(x_{i_m}) \cdots p_n(x_{i_n})] \\ &= w^* - \lim_{l \rightarrow \infty} E[p_{l,1}(x_{i_1}) \cdots p_{l,m}(x_{i_m}) \cdots p_{l,n}(x_{i_n})] \\ &= w^* - \lim_{l \rightarrow \infty} E[p_{l,1}(x_{i_1}) \cdots E[p_{l,m}(x_{i_m})] \cdots p_{l,n}(x_{i_n})] \\ &= E[p_1(x_{i_1}) \cdots E[p_m(x_{i_m})] \cdots p_n(x_{i_n})], \end{aligned}$$

the last equality follows E 's WOT continuity. □

7. MAIN THEOREM AND EXAMPLES

7.1. non-unital tail algebra case. Now, we can state and prove our main theorem for the non-unital tail algebra case:

Theorem 7.1. *Let (\mathcal{A}, ϕ) be a W^* -probability space and $(x_i)_{i \in \mathbb{N}}$ be an infinite sequence of selfadjoint random variables. Suppose \mathcal{A} is the WOT closure of the non-unital algebra generated by $(x_i)_{i \in \mathbb{N}}$ and ϕ is non-degenerated. Then the following statements are equivalent:*

- The joint distribution of $(x_i)_{i \in \mathbb{N}}$ satisfies the invariance conditions associated with the linear coactions of the quantum semigroups $\mathcal{B}_s(n)$'s.*
- The sequence $(x_i)_{i \in \mathbb{N}}$ is identically distributed and boolean independent with respect to a ϕ -preserving normal conditional expectation E onto the non-unital tail algebra \mathcal{T} of the sequence $(x_i)_{i \in \mathbb{N}}$*

Proof. $a) \Rightarrow b)$: By choosing $m = 1$ in proposition 6.16, we have

$$\begin{aligned} & E[p_1(x_{i_1}) \cdots p_2(x_{i_2}) \cdots p_n(x_{i_n})] \\ &= E[E[p_1(x_{i_1})]p_2(x_{i_2}) \cdots p_n(x_{i_n})] \\ &= E[p_1(x_{i_1})]E[p_2(x_{i_2}) \cdots p_n(x_{i_n})] \\ &\quad \dots \\ &= E[p_1(x_{i_1})]E[p_2(x_{i_2})] \cdots E[p_n(x_{i_n})], \end{aligned}$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $p_1, \dots, p_n \in \mathcal{T}\langle X \rangle_0$

$b) \Rightarrow a)$ is a special case of theorem 5.3 \square

7.2. Unital tail algebra case. Let (\mathcal{A}, ϕ) be a W^* probability space with a non-degenerated normal state ϕ and $(x_i)_{i \in \mathbb{N}}$ be a sequence of selfadjoint random variables. Suppose \mathcal{A} is the WOT closure of the unital algebra generated $(x_i)_{i \in \mathbb{N}}$ and ϕ is non-degenerated. Again, we denote by \mathcal{A}_0 the non-unital algebra generated by $(x_i)_{i \in \mathbb{N}}$. Let $I_{\mathcal{A}}$ be the unit of \mathcal{A} , we have considered the case that $1_{\mathcal{A}}$ is contained in $\overline{\mathcal{A}_0}^{w*}$. If $I_{\mathcal{A}}$ is not contained in $\overline{\mathcal{A}_0}^{w*}$, denote by I_1 the unit of $\overline{\mathcal{A}_0}^{w*}$. Then

$$I_2 = I_{\mathcal{A}} - I_1 \neq 0$$

and

$$\mathcal{A} = \mathbb{C}I_2 \oplus \overline{\mathcal{A}_0}^{w*}.$$

For all $x \in \overline{\mathcal{A}_0}^{w*}$, we have

$$I_2 x = (I_{\mathcal{A}} - I_1)x = 0.$$

Let $a \in \overline{\mathcal{A}_0}^{w*}$ such that $\phi(xay) = 0$ for all $x, y \in \overline{\mathcal{A}_0}^{w*}$. For $\bar{x}, \bar{y} \in \mathcal{A}$, there exist two constants $c_1, c_2 \in \mathbb{C}$ and $x, y \in \overline{\mathcal{A}_0}^{w*}$ such that $\bar{x} = c_1 I_2 + x$ and $\bar{y} = c_2 I_2 + y$, then

$$\phi(\bar{x}\bar{y}) = \phi(xab) = 0,$$

Since our \bar{x}, \bar{y} are chosen arbitrarily, we have $a = 0$. Therefore, $(\overline{\mathcal{A}_0}^{w*}, \frac{1}{\phi(I_1)}\phi)$ is a W^* -probability space with a non-degenerated normal state. Let \mathcal{A}_{tail} be the unital tail algebra of $(x_i)_{i \in \mathbb{N}}$ in (\mathcal{A}, ϕ) and \mathcal{T} be the non-unital tail algebra of $(x_i)_{i \in \mathbb{N}}$ in $(\overline{\mathcal{A}_0}^{w*}, \frac{1}{\phi(I_1)}\phi)$. Then, we have

$$\mathcal{A}_{tail} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\} = \bigcap_{n=1}^{\infty} (W^*\{x_k | k \geq n\} + \mathbb{C}I_{\mathcal{A}}) = \mathcal{T} + \mathbb{C}I_{\mathcal{A}}.$$

Since $\overline{\mathcal{A}_0}^{w*}$ is a two-sided ideal of \mathcal{A} . For $\bar{x} \in \mathcal{A}_{tail}$, $\bar{x} = aI_{\mathcal{A}} + x$ for some $x \in \mathcal{T}$ and $a \in \mathbb{C}$. By theorem 7.2, there is a ϕ preserving normal conditional expectation E from $\overline{\mathcal{A}_0}^{w*}$ onto \mathcal{T} . As we proceeded in section 7, we can extend this conditional expectation E to an conditional expectation \bar{E} which is from the unitalization of $\overline{\mathcal{A}_0}^{w*}$ to the unitalization of \mathcal{T} . The unitalizations of the two algebras are isomorphic to \mathcal{A} and \mathcal{A}_{tail} , respectively. We have

Lemma 7.2. *The conditional expectation \bar{E} is ϕ -preserving and normal.*

Proof. The normality is obvious, we just check the ϕ -preserving condition here. Let $\bar{x} = aI_{\mathcal{A}} + x \in \mathcal{A}$ for some $x \in \overline{\mathcal{A}_0}^{w*}$ and $a \in \mathbb{C}$, we have

$$\phi(E[\bar{x}]) = \phi(E[aI_{\mathcal{A}} + x]) = \phi(aI_{\mathcal{A}} + E[x]) = a + \phi(E[x]) = a + \phi(x).$$

The last equality follows the fact that E is a $\phi(I_1)\phi$ -preserving conditional expectation in $(\overline{\mathcal{A}_0}^{w*}, \frac{1}{\phi(I_1)}\phi)$. \square

Together with proposition 5.3 We have the following theorem for our unital case:

Theorem 7.3. *Let (\mathcal{A}, ϕ) be a W^* -probability space and $(x_i)_{i \in \mathbb{N}}$ be a sequence of self-adjoint random variables. Suppose the unit $I_{\mathcal{A}}$ of \mathcal{A} is not contained in the WOT closure of the non-unital algebra generated by $(x_i)_{i \in \mathbb{N}}$ and ϕ is non-degenerated. Then the following statements are equivalent:*

- a) *The joint distribution of $(x_i)_{i \in \mathbb{N}}$ satisfies the invariance condition associated with the linear coactions of the quantum semigroups $\mathcal{B}_s(n)$'s.*
- b) *The sequence $(x_i)_{i \in \mathbb{N}}$ is identically distributed and boolean independent with respect to a ϕ -preserving normal conditional expectation E onto the unital tail algebra \mathcal{A}_{tail} of the $(x_i)_{i \in \mathbb{N}}$.*

7.3. Examples. To illustrate theorem 7.1 and theorem 7.3, we provide two examples here. For the details of the examples, see [4] and [8].

Non-unital case Let \mathcal{H} be a Hilbert space with orthonormal basis $\{e_i\}_{i \in \mathbb{N} \cup \{0\}}$, we define a sequence of operators $\{x_n\}_{n \in \mathbb{N}}$ as follows:

$$x_n e_0 = e_n, \text{ and } x_n e_i = \delta_{n,i} e_0 \text{ for } i \in \mathbb{N}.$$

Let \mathcal{A} be the von Neumann algebra generated by $\{x_n\}_{n \in \mathbb{N}}$, then e_0 is cyclic for \mathcal{A} . Since \mathcal{A} is WOT closed and contains all finite-rank operators, \mathcal{A} is actually $B(\mathcal{H})$. Let ϕ be the vector state $\phi(\cdot) = \langle \cdot e_0, e_0 \rangle$, then we can easily check that the random variables x_i 's are identically distributed and boolean independent. The tail algebra is $\mathbb{C}P_{e_0}$ which does not contain the unit of $B(\mathcal{H})$. The conditional expectation E is given by the following formula:

$$E[x] = P_{e_0} x P_{e_0},$$

for all $x \in \mathcal{A}$.

Unital case Let $\mathcal{H}_1 = \mathcal{H} \oplus \mathbb{C}e_{-1}$ be the direct sum of the Hilbert space \mathcal{H} with orthonormal basis $\{e_i\}_{i \in \mathbb{N} \cup \{0\}}$ and $\mathbb{C}P_{e_{-1}}$. As we constructed in the previous example, we define a sequence of operators $\{x_n\}_{n \in \mathbb{N}}$ as follows:

$$x_n e_0 = e_n, \text{ and } x_n e_i = \delta_{n,i} e_0 \text{ for } i \in \mathbb{N}.$$

Let \mathcal{A} be the von Neumann algebra generated by $\{x_n\}_{n \in \mathbb{N}}$, then $\mathcal{A} = B(\mathcal{H}) \oplus \mathbb{C}P_{e_{-1}}$. Therefore, the WOT-closure of the non-unital algebra generated by $\{x_n\}_{n \in \mathbb{N}}$ is $B(\mathcal{H}) \oplus 0$ but not the entire algebra \mathcal{A} . Let ϕ be the vector state $\phi(\cdot) = \frac{1}{2} \langle \cdot (e_0 + e_{-1}), e_0 + e_{-1} \rangle$, then the random variables x_i 's are identically distributed and boolean independent. The unital tail algebra is $\mathbb{C}I_{\mathcal{H}} \oplus \mathbb{C}P_{e_0}$ which contains the unit of $B(\mathcal{H}_1)$. The conditional expectation E is given by the following formula:

$$E[x] = P_{e_0} x P_{e_0} + \langle x e_{-1}, e_{-1} \rangle (I_{\mathcal{H}_1} - P_{e_{-1}}),$$

for all $a \in \mathcal{A}$.

7.4. On W^* -probability spaces with faithful states. If we restrict the invariance condition for boolean independence to a W^* -probability space with a faithful state, then we will have the following:

Theorem 7.4. *Let (\mathcal{A}, ϕ) be a W^* -probability space and $(x_i)_{i \in \mathbb{N}}$ be a sequence of self-adjoint random variables such that \mathcal{A} is generated by $(x_i)_{i \in \mathbb{N}}$ and ϕ is faithful. Then the following statements are equivalent:*

- a) *The joint distribution of $(x_i)_{i \in \mathbb{N}}$ satisfies the invariance condition associated with the linear coactions of the quantum semigroups $\mathcal{B}_s(n)$'s.*
- b) *$x_i = x_j$ for all $i, j \in \mathbb{N}$*

Proof. b) \Rightarrow a): If $x_i = x_j$ for all $i, j \in \mathbb{N}$, given a monomial $p = X_{i_1} \cdots X_{i_k} \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, then

$$\begin{aligned} \mu_{x_1, \dots, x_n}(X_{i_1} \cdots X_{i_k})\mathbf{P} &= \phi(x_{i_1} \cdots x_{i_k})\mathbf{P} \\ &= \phi(x_1^k)\mathbf{P} \\ &= \sum_{j_1, \dots, j_k=1}^n \phi(x_1^k) \pi(\mathbf{P} u_{j_1, i_1} \cdots u_{j_k, i_k} \mathbf{P}) \\ &= \sum_{j_1, \dots, j_k=1}^n \phi(x_{j_1} \cdots x_{j_k}) \mathbf{P} u_{j_1, i_1} \cdots u_{j_k, i_k} \mathbf{P} \\ &= \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}(\mathbb{L}p). \end{aligned}$$

b) \Rightarrow a): It is sufficient to show that $x_1 = x_2$. By theorem 7.1 and 7.3, there exists a ϕ -preserving conditional expectation E maps \mathcal{A} to its unital or non-unital tail algebra such that $(x_i)_{i \in \mathbb{N}}$ is identically boolean independent with respect to E . For $k \in \mathbb{N}$ and $k > 2$, we have

$$\begin{aligned} &\phi((x_1 - x_2)x_k((x_1 - x_2)x_k)^*) \\ &= \phi((x_1 - x_2)x_k^2(x_1 - x_2)) \\ &= \phi(E[(x_1 - x_2)x_k^2(x_1 - x_2)]) \\ &= \phi(E[x_1 - x_2]E[x_k^2]E[x_1 - x_2]) \\ &= 0. \end{aligned}$$

Since ϕ is faithful, we get

$$(x_1 - x_2)x_k = 0$$

for all $k > 2$. Let \mathcal{A}_n be the WOT closure of the non-unital algebra generated by $\{x_k | k > n\}$, then we have

$$(x_1 - x_2)x = 0$$

for all $x \in \mathcal{A}_k$. Notice that $(x_i)_{i \in \mathbb{N}}$ is exchangeable, by the construction of proposition 4.2 in [15], there exists a normal ϕ -preserving homomorphism $\alpha : \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}$ such that $\alpha(x_i) = x_{i+1}$. Denote by I_n the unit of \mathcal{A}_n , then $\alpha(I_n) = I_{n+1}$ and $I_n I_{n+1} = I_{n+1}$, since I_{n+1} is a projection in \mathcal{A}_n . Then, we have

$$\phi((I_n - I_{n+1})^2) = \phi(I_n - I_{n+1}) = \phi(I_n) - \phi(\alpha(I_n)) = 0,$$

which implies that $I_n = I_{n+1}$. It follows that

$$I_0 = I_1 = I_2.$$

Therefore,

$$0 = (x_1 - x_2)I_2 = (x_1 - x_2)I_0 = x_1 - x_2.$$

□

8. TWO MORE KINDS OF DISTRIBUTIONAL SYMMETRIES

Since $\mathbb{C}\langle X_1, \dots, X_n \rangle$ is an algebra which is freely generated by n indeterminants X_1, \dots, X_n . It would be natural to define coactions of the quantum semigroups $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \dots, X_n \rangle$ as an algebraic homomorphism but not only a linear map. In this section, we will study the probabilistic symmetries associated with some algebraic coactions of the quantum semigroups $\mathcal{B}_s(n)$'s and $B_s(n)$'s on $\mathbb{C}\langle X_1, \dots, X_n \rangle$. We will define the invariance condition for the joint distribution of a sequence of noncommutative random variables in a similar form as we did in previous sections.

Now, let us consider $\mathbb{C}\langle X_1, \dots, X_n \rangle$ as an algebra and define a coaction of the quantum semigroups $\mathcal{B}_s(n)$ to be a homomorphism

$$\mathbb{L}'_n : \mathbb{C}\langle X_1, \dots, X_n \rangle \rightarrow \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathcal{B}_s(n)$$

by the following formulas:

$$\mathbb{L}'_n(1) = 1 \otimes I, \quad \mathbb{L}'_n(X_i) = \sum_{k=1}^n X_k \otimes \mathbf{P}u_{k,i} \mathbf{P}.$$

Then, we would have

$$L_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1, \dots, j_k=1}^n X_{j_1} \cdots X_{j_k} \otimes \mathbf{P}u_{j_1, i_1} \mathbf{P} \cdots \mathbf{P}u_{j_k, i_k} \mathbf{P}$$

and

$$(\mathbb{L}'_n \otimes id_{\mathcal{B}_s(n)})\mathbb{L}'_n = (id_{\mathbb{C}_n} \otimes \Delta)\mathbb{L}'_n.$$

We will call \mathbb{L}'_n the algebraic coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \dots, X_n \rangle$. The invariance condition is so strong that we can get our conclusion in some finitely generated probability spaces.

Proposition 8.1. *Let (\mathcal{A}, ϕ) be a W^* -probability space with a non-degenerated state ϕ , Fixed $n \in \mathbb{N}$, let $(x_i)_{i=1, \dots, n}$ be a sequence of selfadjoint noncommutative random variables in \mathcal{A} . We say the joint distribution of $(x_i)_{i=1, \dots, n}$ is invariant under the algebraic coaction \mathbb{L}'_n of $\mathcal{B}_s(n)$ if*

$$\mu_{x_1, \dots, x_n}(p)\mathbf{P} = \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}(\mathbb{L}'_n(p)),$$

for all $p \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, where μ_{x_1, \dots, x_n} is the joint distribution of $(x_i)_{i=1, \dots, n}$. If is the WOT closure of the unital algebra generated by $(x_i)_{i=1, \dots, n}$, then the joint distribution of $(x_i)_{i=1, \dots, n}$ is invariant under the algebraic coaction \mathbb{L}'_n of $\mathcal{B}_s(n)$ is equivalent to $x_1 = x_2 = \cdots = x_n$.

Proof. Suppose $x_1 = x_2 = \cdots = x_n$. Let $p = X_{i_1} \cdots X_{i_m} \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, then we have

$$\begin{aligned} & \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}(\mathbb{L}'_n(X_{i_1} \cdots X_{i_m})) \\ &= \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}\left(\sum_{j_1, \dots, j_m=1}^n X_{j_1} \cdots X_{j_m} \otimes \mathbf{P}u_{j_1, i_1} \mathbf{P}u_{j_2, i_2} \mathbf{P} \cdots \mathbf{P}u_{j_m, i_m} \mathbf{P}\right) \\ &= \sum_{j_1, \dots, j_m=1}^n \phi(x_{j_1} \cdots x_{j_m}) \mathbf{P}u_{j_1, i_1} \mathbf{P}u_{j_2, i_2} \mathbf{P} \cdots \mathbf{P}u_{j_m, i_m} \mathbf{P} \\ &= \sum_{j_1, \dots, j_m=1}^n \phi(x_1^m) \mathbf{P}u_{j_1, i_1} \mathbf{P}u_{j_2, i_2} \mathbf{P} \cdots \mathbf{P}u_{j_m, i_m} \mathbf{P} \\ &= \phi(x_1^m) \mathbf{P} \\ &= \mu_{x_1, \dots, x_n}(X_{i_1} \cdots X_{i_m}) \mathbf{P}. \end{aligned}$$

Since p is arbitrary, we proved \Leftarrow .

Suppose the joint distribution of $(x_i)_{i=1,\dots,n}$ is invariant under the algebraic coaction \mathbb{L}'_n . Let $\{v_1, \dots, v_{2n}\}$ be orthonormal basis of the standard $2n$ -dimensional Hilbert space \mathbb{C}^{2n} and denote $v_k = v_{k+2n}$ for all $k \in \mathbb{Z}$. Let

$$P_{i,j} = P_{v_{2(i-j)+1} + v_{2(j-i)+2}}$$

and

$$P = P_{v_1 + v_2 + \dots + v_{2n}},$$

where P_v is the orthogonal projection onto the one dimensional subspace generated by the vector v . By lemma 3.3, we have a representation π of $\mathcal{B}_s(n)$ on \mathbb{C}^{2n} defined by the following formulas:

$$\pi(\mathbf{P}) = P, \quad \pi(\mathbf{P}u_{i_1,j_1} \cdots u_{i_k,j_k} \mathbf{P}) = PP_{i_1,j_1} \cdots P_{i_k,j_k} P$$

for all $i_1, j_1, \dots, i_k, j_k \in \{1, \dots, n\}$ and $k \in \mathbb{N}$. In particular, we have

$$\pi(\mathbf{P}u_{i,j} \mathbf{P}) = PP_{i,j}P = \frac{1}{n}P.$$

Let π act on the invariance condition, we get

$$\begin{aligned} \phi(x_{i_1} \cdots x_k)P &= \pi(\mu_{x_1, \dots, x_n}(X_{i_1} \cdots X_{i_k})\mathbf{P}) \\ &= \pi\left(\sum_{j_1, \dots, j_k=1}^n \mu_{x_1, \dots, x_n} \otimes id_{\mathcal{B}_s(n)}(X_{j_1} \cdots X_{j_k} \otimes \mathbf{P}u_{j_1,i_1} \mathbf{P} \cdots \mathbf{P}u_{j_k,i_k} \mathbf{P})\right) \\ &= \sum_{j_1, \dots, j_k=1}^n \phi(x_{j_1} \cdots x_{j_k})\pi(\mathbf{P}u_{j_1,i_1} \mathbf{P} \cdots \mathbf{P}u_{j_k,i_k} \mathbf{P}) \\ &= \sum_{j_1, \dots, j_k=1}^n \phi(x_{j_1} \cdots x_{j_k})\frac{1}{n^k}P, \end{aligned}$$

for all $i_1, i_k \in \{1, \dots, n\}$. It implies that

$$\phi(x_{i_1} \cdots x_k) = \frac{1}{n^k} \sum_{j_1, \dots, j_k=1}^n \phi(x_{j_1} \cdots x_{j_k}).$$

Therefore, two mixed moments are the same if their degree are the same. Given two monomials $a = x_{s_1} \cdots x_{s_{k_1}}$ and $b = x_{s_1} \cdots x_{s_{k_2}}$ then

$$\phi(a(x_i - x_j)(x_i - x_j)^*b) = \phi(a(x_i - x_j)^2b) = \phi(ax_i x_i b) - \phi(ax_i x_j b) - \phi(ax_i x_j b) + \phi(ax_i x_i b) = 0,$$

the last equation is true because all the monomials have the same degree. By some linear combinations, we have

$$\phi(a(x_i - x_j)(x_i - x_j)^*b) = 0,$$

for all $a, b \in \mathcal{A}_{[n]}$, where $\mathcal{A}_{[n]}$ is the unital algebra generated by x_1, \dots, x_n , thus

$$x_i = x_j,$$

for all $i \neq j$. □

In the end of this section, we study a coaction of the quantum semigroups $B_s(n)$ on the joint distribution of a sequence of noncommutative random variables. We can define a coaction

$$L_n : \mathbb{C}\langle X_1, \dots, X_n \rangle \rightarrow \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes B_s(n)$$

of $B_s(n)$ on the algebra of noncommutative polynomials $\mathbb{C}\langle X_1, \dots, X_n \rangle$ by the following formulas:

$$L_n(1) = 1 \otimes I, \quad L_n(X_i) = \sum_{k=1}^n X_k \otimes u_{k,i},$$

where 1 is the identity of $\mathbb{C}\langle X_1, \dots, X_n \rangle$ and I is the identity of $B_s(n)$. According to the definition of L_n , we have

$$L_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1, \dots, j_k=1}^n X_{j_1} \cdots X_{j_k} \otimes u_{j_1, i_1} \cdots u_{j_n, i_n}.$$

One can easily check

$$(L_n \otimes id_{B_s(n)})L_n = (id_{\mathbb{C}_n} \otimes \Delta)L_n,$$

where $id_{B_s(n)}$ and $id_{\mathbb{C}_n}$ are identity map of $B_s(n)$ and $\mathbb{C}\langle X_1, \dots, X_n \rangle$.

Then, we have

Proposition 8.2. *Let (\mathcal{A}, ϕ) be a probability space and $(x_i)_{i \in \mathbb{N}}$. The joint distribution is invariant under the coactions L_n 's of the quantum semigroups $B_s(n)$ if for all n , $p \in \mathbb{C}\langle X_1, \dots, X_n \rangle$. Let μ_{x_1, \dots, x_n} be the joint distribution of x_1, \dots, x_n , I be the unit of $B_s(n)$ and L_n is the coaction on $\mathbb{C}\langle X_1, \dots, X_n \rangle$, we have*

$$\mu_{x_1, \dots, x_n}(p)I = \mu_{x_1, \dots, x_n} \otimes id(L_n(p)).$$

If the joint distribution of $(x_i)_{i \in \mathbb{N}}$ is invariant under the coactions L_n 's of the quantum semigroups $B_s(n)$'s, then $\phi(x_{i_1} x_{i_2} \cdots x_{i_k}) = 0$ for all $i_1, \dots, i_k \in \mathbb{N}$ and $k \in \mathbb{N}$.

Proof. Let k be a positive integer, $i_1, \dots, i_k \in \mathbb{N}$ and $N = \max\{i_1, \dots, i_k\}$. Take a trivial representation π of $B_s(N)$ on a 1-dimensional space V defined by the following formulas:

$$\pi(I) = 1, \quad \text{and} \quad \pi(u_{i,j}) = \pi(\mathbf{P}) = 0,$$

where 1 is the identity of V . By the universality of $B_s(n)$, π is well defined. According to the invariance condition, we have

$$\pi(\mu_{x_1, \dots, x_N}(p)I) = \mu_{x_1, \dots, x_N} \otimes \pi(L_n(p)),$$

for all $p \in \mathbb{C}\langle X_1, \dots, X_n \rangle$. Let $p = X_{i_1} \cdots X_{i_k}$, we get

$$\phi(x_{i_1} x_{i_2} \cdots x_{i_k})1 = \sum_{j_1, \dots, j_k=1}^n \phi(x_{j_1} \cdots x_{j_k})\pi(u_{j_1, i_1} \cdots u_{j_n, i_n}) = 0,$$

which completes the proof. \square

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